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Tests for Einstein-Podolsky-Rosen steering in two-mode systems of identical massive bosons

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In a previous paper tests for entanglement for two-mode systems involving identical massive bosons were obtained. In the present paper we consider sufficiency tests for Einstein-Podolsky-Rosen (EPR) steering in such systems. We find that spin squeezing in any spin component, a Bloch vector test, the Hillery-Zubairy planar spin variance test, and squeezing in two-mode quadratures all show that the quantum state is EPR steerable. We also find a generalization of the Hillery-Zubairy planar spin variance test for EPR steering. The relation to previous correlation tests is discussed. This paper is based on a detailed classification of quantum states for bipartite systems. States for bipartite composite systems are categorized in quantum theory as either separable or entangled, but the states can also be divided differently into Bell local or Bell nonlocal states in terms of local hidden variable theory (LHVT). For the Bell local states there are three cases depending on whether both, one of or neither of the LHVT probabilities for each subsystem are also given by a quantum probability involving subsystem density operators. Cases where one or both are given by a quantum probability are known as local hidden states (LHSs) and such states are nonsteerable. The steerable states are the Bell local states where there is no LHS, or the Bell nonlocal states. The relationship between the quantum and hidden variable theory classification of states is discussed.

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I. INTRODUCTION

Recent papers by Dalton *et al.* [1–3] have dealt with the topic of *bipartite quantum entanglement* and experimental tests for its demonstration in the context of two-mode systems of *identical massive bosons*. However, although the quantum states of composite systems can just be classified into disjoint sets of *separable* or *entangled* states, it is also possible to classify them into distinct categories based on *local hidden variable theory* [4], where the two basic disjoint subsets of quantum states are now the *Bell local* states and the *Bell nonlocal* states. The latter categorization is based on whether or not the probability $P(a, b|A, B, c)$ for measured outcomes a, b on subsystem observables A, B for state preparation process c, is given by a local hidden variable theory (LHVT) form $P(a, b|A, B, c) = \sum_{\lambda} P(\lambda|c)P(a|A, c, \lambda)P(b|B, c, \lambda)$ (where preparation c results in a probability distribution $P(\lambda|c)$ for hidden variables λ, $P(a|A, c, \lambda)$ is the probability for measured outcome a on subsystem observable$A$ when the hidden variables are λ with $P(b|B, c, \lambda)$ the analogous observable B probability). Quantum states where $P(a, b|A, B, c)$ is given by a LHVT form are Bell local; if not they are Bell nonlocal and associated with Bell inequality violation experiments. Hence, in accord with the idea set out in the Einstein-Podolsky-Rosen (EPR) paper [5] that the predictions based on quantum theory could also be the statistical outcome of an underlying deterministic theory (involving what we now would regard as hidden variables), the predictions based on the local hidden variable theory (the Bell local states) will be regarded as being in agreement with quantum theory—and the relevant expressions will be interchangeable. The Bell nonlocal states will be those quantum states where the local HVT does not apply, and there is no underlying deterministic theory that leads to the quantum results. However, within the Bell local states a further categorization is possible which is relevant to whether EPR steering occurs. Based on the concept of *local hidden states* (LHSs) introduced by Wiseman *et al.* [6–8], we show that the Bell local states for *bipartite* systems can be divided into three *disjoint* subcategories, with a fourth corresponding to the Bell nonlocal states. These four categories of states associated with local hidden variable theory have differing features regarding entanglement, EPR steering and Bell nonlocality—as will be explained below (see also Ref. [9]). For readers unfamiliar with the hidden variable theory issue and LHSs, a brief overview is presented in Appendix A, emphasizing the key papers of Einstein, Schrödinger, Bell, and Werner [4,5,10–12] and those of Wiseman *et al.* [6–8].

The present paper is one of a series aimed at developing tests based on experimentally measurable quantities that are sufficient (though not necessary) for determining which category applies for specific quantum states of bipartite two-mode systems of identical massive bosons. The focus of the present paper is on sufficiency tests for demonstrating *EPR steering* in these systems—essentially by eliminating two of the four possible categories of quantum states. We find that spin squeezing in any spin component, a Bloch vector test, the Hillery-Zubairy planar spin variance test, and squeezing in two-mode quadratures are all sufficiency tests to show that the

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quantum state is EPR steerable. In addition, a generalization of the Hillery-Zubairy planar spin variance test for EPR steering is also found. Apart from the two planar spin variance tests, the tests depend on applying the local particle number super-selection rule (SSR).

The plan of the paper is as follows. In Sec. II we begin by first presenting the *quantum theory expressions* for joint and single measurement probabilities for *bipartite quantum systems*, and then the possible underlying *local hidden variable theory* (LHVT) expressions. Only *von Neumann* measurements will be considered. In accordance with the requirement that HVT does not give different experimental predictions, the quantum expressions (1), (2), and (3) will be regarded as *always* applying—irrespective of additional local hidden variable theory formulas that apply *as well*. In the present paper, for quantum theory the preparation process is reflected in the *density operator* for the system. In HVT the preparation process is reflected in the *probability function* for the hidden variables. We restrict LHVT to a version where the measurement outcomes for the observables in LHVT are the same as the possible quantum theory outcomes, determined as the *eigenvalues* of the corresponding quantum *Hermitian operators*. For simplicity we treat the outcomes as *quantized*—the generalization for *continuous* eigenvalues is straightforward. Important relationships between the probabilities and mean values for measurements given by quantum theory and by local hidden variable theory are highlighted. This linkage does not of course apply for Bell nonlocal states. The issue of interrelating the Hermitian operators and c-number variables that describe the same observable is nontrivial and is described in Sec. IV for the specific two-mode system of interest. Although LHVT does not have one unique form, we must choose a version such that its predictions agree with those from quantum theory. There would be no point in considering a LHVT that was *not* in agreement with quantum theory! A *key point* is that because LHVT *underlies* quantum theory, *any* result we establish for mean values, variances of observables using LHVT for a quantum state that is *also* Bell local, can immediately be expressed in terms of the equivalent Hermitian *operators* that describe the same observables, together with the quantum *density operator* that specifies the *same* state instead of the set of LHVT *probabilities*. Obviously, it is also important to consider how to interrelate the Hermitian operators that represent observables in quantum theory with the c-number quantities representing the same observables in LHVT. General features for joint and single measurement probabilities are set out in Appendix B.

In Sec. III we then consider the detailed description of how the quantum states for bipartite systems may be categorized. We relate our categories of states to the hierarchy of subsets discussed in Refs. [6–8,13].

In Sec. IV various tests for *EPR steering* are considered for the case where each subsystem consists of a *single mode* and the particles that occupy it are *massive bosons*, taking into account that the *local hidden states* must comply with the local particle number *super-selection rule* (see Refs. [1–3]) since they must be possible quantum states for the particular subsystem considered on its own. The question of how to relate the quantum Hermitian operators to the LHVT c-number variables that describe the same observables is dealt with in this section. Since mode annihilation and creation operators are not Hermitian we can replace these by *quadrature operators*, including in expressions for spin operators and other important quantities. In applying LHVT the quadrature operators are replaced by c-number quadrature amplitudes. However, in order to achieve a *reciprocal interconversion* between the Hermitian operators and the c-number variables that represent the same observable, it has been necessary to introduces certain additional *auxiliary observables* and allow the c-number versions of these to have their own LHVT probability distributions. This seems to be the best version of LHVT to ensure that the quantum theory and the LHVT are describing the same physical measurements. It turns out that previous sufficiency tests (see Refs. [1–3] for details) for quantum entanglement (*Bloch vector* test, *spin squeezing* in any spin component S_x, S_y, or S_z, the Hillery-Zubairy *planar spin variance* test [14], a *two-mode quadrature squeezing* test) can *also* be applied as sufficiency tests for EPR steering in two-mode systems of identical massive bosons. However, in addition a *different* planar spin variance test for EPR steering involving the sum of the variances for spin operators S_x, S_y and the mean boson number has been obtained here which also involves the mean value for S_z, generalizing a result in He *et al.* [15]. This test is a generalization of the Hillery-Zubairy planar spin variance test. In addition there are *weak* and *strong correlation* tests for EPR steering that have been previously obtained by Cavalcanti *et al.* [16]. However, as each of the correlation tests are equivalent to some of the other tests, we include these in the Appendices rather than in the main body of the paper. The two planar spin variance tests can also be proved without applying the local particle number SSR. However, for convenience we include the proofs for these tests within Sec. IV, as well as covering in Appendices I and J the non-SSR-dependent proofs based on the correlation tests in Ref. [16]. Section V provides a summary of the main results. An illustration of applying the EPR tests is given for the case of the two-mode *binomial* state—which is shown to be EPR steerable.

In Sec. IV we will identify experiments demonstrating EPR steering in *two-mode* Bose-Einstein condensates according to these tests, such as in Refs. [17–22] that have already been carried out, though EPR steering was identified only in Refs. [21] and [22]. Note also that EPR steering has also recently been found in three- and four-mode systems [23–25] based on different tests (such as in Ref. [26]) for these multimode cases. The test in Ref. [23] for verifying EPR steering involves direct measurement tests on variances of conjugate observables for one subsystem, to see whether the Heisenberg uncertainty principle has been violated after measurements were made on the other subsystem.

Details are set out in Appendices. Appendix A presents a brief summary of the *development* of hidden variable theory and contains an overview of the *categorization* of quantum states both as separable or entangled on the one hand or as Bell local and Bell nonlocal on the other, pointing out that Bell local states may be further subcategorized in terms of the presence or otherwise of LHSs, as introduced by Wiseman *et al.* Appendix B sets out the general relations for measurement probabilities in bipartite systems. In Appendix C general properties of mean values and variances are reviewed.

Expressions for classical observables in terms of quadrature amplitudes are given in Appendix D. The Werner states are described in Appendix E, since in various parameter regimes they provide examples of the four categories of states in the local hidden variable theory model. The idea behind EPR steering is discussed in Appendix F. Details for the derivation of the spin squeezing and two-mode quadratures EPR steering tests are presented in Appendices G and H, The correlation tests and their forms in terms of spin operators are set out in Appendices I and J.

II. MEASUREMENT PROBABILITIES IN BIPARTITE SYSTEMS

In this section we set out the expressions for joint and single measurement probabilities for bipartite systems, both in quantum theory and in local hidden variable theory. Based on Einstein's view that quantum theory is underpinned by LHVT, the relationship between the two approaches is also pointed out. General results for the probabilities are set out in Appendix B. The same *notation* for observables, their measured outcomes and the measurement probabilities will be used for both the quantum theory and LHVT situations.

A. Quantum theory—Measurement probabilities

In *quantum theory* the *joint probability* $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ for measurement of *any* pair of subsystem *observables* Ω_A and Ω_B to obtain *any* of their possible *outcomes* α and β when the *preparation* process is c is given by an expression based on the subsystem observables Ω_A and Ω_B being represented by quantum *Hermitian operators* $\widehat{\Omega}_A$ and $\widehat{\Omega}_B$. Here simultaneous precise measurement applies because the system operators involved, $\widehat{\Omega}_A\otimes\widehat{1}_B$ and $\widehat{1}_B\otimes\widehat{\Omega}_B$ *commute* and therefore have complete sets of simultaneous eigenvectors.

We have for the *joint measurement probability* [see Ref. [6], Eq. (2)]

$$P(\alpha,\beta|\Omega_A,\Omega_B,c)=\mathrm{Tr}\big(\big(\widehat{\Pi}_\alpha^A\otimes\widehat{\Pi}_\beta^B\big)\widehat{\rho}\big), \tag{1}$$

where $\widehat{\Pi}_\alpha^A$ and $\widehat{\Pi}_\beta^B$ are *projectors* onto the *eigenvector spaces* for $\widehat{\Omega}_A$ and $\widehat{\Omega}_B$ associated with the real *eigenvalues* α and β that in quantum theory are the *possible* measurement outcomes. We have $\widehat{\Omega}_A\widehat{\Pi}_\alpha^A=\alpha\widehat{\Pi}_\alpha^A=\widehat{\Pi}_\alpha^A\widehat{\Omega}_a$, and similar expressions for $\widehat{\Pi}_\beta^B$. Clearly the quantum expression for the joint probability satisfies the general probability requirement (B1) that the sum over all possible outcomes is unity—the sum rules over α and β being implemented via the *projector properties* $\sum_\alpha\widehat{\Pi}_\alpha^A=\widehat{1}^A$ and $\sum_\beta\widehat{\Pi}_\beta^B=\widehat{1}^B$ involving the subsystem *unit operators* and $\mathrm{Tr}\widehat{\rho}=1$.

The quantum theory expressions for the *single measurement probabilities*

$$\begin{aligned}P(\alpha|\Omega_A,c)&=\mathrm{Tr}\big(\big(\widehat{\Pi}_\alpha^A\otimes\widehat{1}^B\big)\widehat{\rho}\big),\\ P(\beta|\Omega_B,c)&=\mathrm{Tr}\big(\big(\widehat{1}^A\otimes\widehat{\Pi}_\beta^B\big)\widehat{\rho}\big),\end{aligned} \tag{2}$$

for (respectively) measuring Ω_A to have outcome α irrespective of Ω_B and β or for measuring Ω_B to have outcome β irrespective of Ω_A and α both follow from (B2) or (B3) and the projector properties. The single measurement probabilities can be expressed in terms of *reduced density operators* $\widehat{\rho}^A$ and $\widehat{\rho}^B$ for the subsystems

$$\begin{aligned}\widehat{\rho}^A&=\mathrm{Tr}_B(\widehat{\rho}),\quad P(\alpha|\Omega_A,c)=\mathrm{Tr}_A\big(\widehat{\Pi}_\alpha^A\widehat{\rho}^A\big),\\ \widehat{\rho}^B&=\mathrm{Tr}_A(\widehat{\rho}),\quad P(\beta|\Omega_B,c)=\mathrm{Tr}_B\big(\widehat{\Pi}_\beta^B\widehat{\rho}^B\big).\end{aligned} \tag{3}$$

The proof of the results (3) for $P(\alpha|\Omega_A,c)$ and $P(\beta|\Omega_B,c)$ is straightforward. Note that in general the reduced density operators require first knowing the *overall* system density operator $\widehat{\rho}$. The joint and single measurement probabilities are related via (B3) and (B2), as easily shown using $\sum_\alpha\widehat{\Pi}_\alpha^A=\widehat{1}^A$ and $\sum_\beta\widehat{\Pi}_\beta^B=\widehat{1}^B$. Using similar considerations and $\mathrm{Tr}\widehat{\rho}=1$, the single measurement probabilities also satisfy the sum rules (B4).

The *conditional probabilities* are given by the general expressions (B5) that apply for both quantum and LHVT cases.

The *mean value* for joint measurement outcomes of the observables $\widehat{\Omega}_A$ and $\widehat{\Omega}_B$ will be given by

$$\begin{aligned}\langle\widehat{\Omega}_A\otimes\widehat{\Omega}_B\rangle&=\sum_{\alpha,\beta}\alpha\,\beta\,P(\alpha,\beta|\Omega_A,\Omega_B,c)\\ &=\mathrm{Tr}(\widehat{\Omega}_A\otimes\widehat{\Omega}_B)\widehat{\rho},\end{aligned} \tag{4}$$

where the results $\sum_\alpha\alpha\,\widehat{\Pi}_\alpha^A=\widehat{\Omega}_A$ and $\sum_\beta\beta\,\widehat{\Pi}_\beta^B=\widehat{\Omega}_B$ and (1) have been used.

The mean value for the measurement of a single observable $\widehat{\Omega}_A$ is

$$\begin{aligned}\langle\widehat{\Omega}_A\rangle&=\sum_\alpha\alpha\,P(\alpha|\Omega_A,c)=\mathrm{Tr}(\widehat{\Omega}_A\otimes\widehat{1}_B)\widehat{\rho}\\ &=\mathrm{Tr}_A(\widehat{\Omega}_A\,\widehat{\rho}^A),\end{aligned} \tag{5}$$

as can be derived from (1) and (3).

It is worth noting that for systems of identical massive bosons *super-selection rules* (SSRs) require the overall density operator $\widehat{\rho}$ to commute with the *total* number operator N (*global* particle number SSR—see, for example, Refs. [2,3] and references therein for discussions of SSRs). Consequently the density operator for a two-mode system

$$\begin{aligned}\widehat{\rho}=&\sum_{n_A,n_B}\sum_{m_A,m_B}\rho(n_A,n_B;m_A,m_B)\\ &\times(|n_A\rangle\otimes|n_B\rangle)(\langle m_A|\otimes\langle m_B|)\end{aligned} \tag{6}$$

is such that $\rho(n_A,n_B;m_A,m_B)=0$ unless $n_A+n_B=m_A+m_B$. It is then straightforward to show that the reduced density operator $\widehat{\rho}^A$ for mode A is given by

$$\widehat{\rho}^A=\sum_{n_A}\left[\sum_{n_B}\rho(n_A,n_B;n_A,n_B)\right](|n_A\rangle\langle n_A|), \tag{7}$$

which is SSR compliant for the *subsystem* particle number N_A (*local* particle number SSR). This feature will turn out to be relevant for evaluating terms associated with the EPR steering tests. Note that in general the reduced density operator $\widehat{\rho}^A$ depends on the full density matrix for *both* subsystems, unlike that for a LHS.

B. Local hidden variable theory—Measurement probabilities

A *hidden variable theory* (HVT) is based on hidden variables λ which describe the *real* or *underlying* state of the system, and which are determined with a *probability* $P(\lambda|c)$ for a preparation process c. The probability $P(\lambda|c)$ is real, positive, and its sum over all possible hidden variables is also unity. Thus

$$\sum_{\lambda} P(\lambda|c) = 1. \tag{8}$$

The preparation process is thus reflected in the *probability function* for the hidden variables $c \rightarrow P(\lambda|c)$. In order to maintain generality, the nature of the hidden variables and what fundamental equations determine them is best left unspecified. We are also ignoring any time delay between preparation of the state and measurements on it, so dynamical evolution of hidden variables during this interval is irrelevant. Discussion of successive measurements is not considered here, so whether the hidden variables change as a result of measurement is also beyond the scope of this paper. The key feature is that having been determined in the preparation process, the hidden variables still determine the outcome probabilities in separated subsystems.

In *local hidden variable theory* the *joint probability* $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ for measurement of *any* pair of subsystem *observables* Ω_A and Ω_B to obtain *any* of their possible *outcomes* α and β when the *preparation* process is c is given by an expression involving measurement probabilities $P(\alpha|\Omega_A,c,\lambda)$ and $P(\beta|\Omega_B,c,\lambda)$ for the *separate* subsystems, and which depend on the hidden variables λ. The subsystem observables Ω_A and Ω_B are represented by *c-numbers* rather than Hermitian operators. Here $P(\alpha|\Omega_A,c,\lambda)$ is the probability that measurement of the *observable* Ω_A of subsystem A results in *outcome* α when the *hidden variable* are λ, with a similar definition for $P(\beta|\Omega_B,c,\lambda)$.

For a *LHVT* the *joint probability* $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ for measurement of *any* pair of subsystem *observables* Ω_A and Ω_B to obtain *any* of their possible *outcomes* α and β when the *preparation* process is c is given by [see Ref. [6], Eq. (3) and Ref. [8], Eq. (15)]

$$P(\alpha,\beta|\Omega_A,\Omega_B,c) = \sum_{\lambda} P(\alpha|\Omega_A,c,\lambda)P(\beta|\Omega_B,c,\lambda)P(\lambda|c). \tag{9}$$

In *LHVT* the *hidden variables* λ are *global* and first determined (probabilistically) via the *preparation* process, but then act *locally* to determine the *subsystem* measurement *probabilities* $P(\alpha|\Omega_A,c,\lambda)$ and $P(\beta|\Omega_B,c,\lambda)$—even in the situation where the subsystems are localized in *well-separated* spatial regions and the two subsystem measurements occur *simultaneously*. The probabilities are then finally combined in accordance with *classical probability theory* to determine the joint measurement probability. States for which the joint probability is given by the local hidden variable theory Eq. (9) are referred to as *Bell local*. States where this does not apply are the *Bell nonlocal* states.

In a *nonlocal hidden variable theory* we would just have $P(\alpha,\beta|\Omega_A,\Omega_B,c) = \sum_{\lambda} P(\alpha,\beta|\Omega_A,\Omega_B,c,\lambda)\,P(\lambda|c)$, with no local subsystem probabilities involved. Here $P(\alpha,\beta|\Omega_A,\Omega_B,c,\lambda)$ is the joint probability that measurement of the *observables* Ω_A, Ω_B, of subsystems A, B results in *outcomes* α, β when the *hidden variables* are represented by λ, and $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ is not factorizable.

For LHVT the subsystem probabilities $P(\alpha|\Omega_A,c,\lambda)$ and $P(\beta|\Omega_B,c,\lambda)$ are *not necessarily* given by quantum expressions such as (2) though they *may* be. Following the approach of Refs. [6,7] we will introduce a *more specific* notation (subscript Q) to distinguish cases where $P(\alpha|\Omega_A,c,\lambda)$ and/or $P(\beta|\Omega_B,c,\lambda)$ *are* given by quantum expressions from those where they are not. When the $P_Q(\gamma|\Omega_C,c,\lambda)$ for subsystem C ($C = A, B$) are determined from a quantum expression which involves a *density operator* $\widehat{\rho}^C(c,\lambda)$ for subsystem C determined from the hidden variables λ, then $\widehat{\rho}^C(c,\lambda)$ specifies a so-called *local hidden state* (LHS).

The *single measurement* probabilities $P(\alpha|\Omega_A,c,\lambda)$ and $P(\beta|\Omega_B,c,\lambda)$ must of course satisfy the general requirements of being *real*, *positive* and such that their *sum* over all possible outcomes is *unity* for each value λ of the LHV in accordance with the general requirements (B4). Thus

$$\sum_{\alpha} P(\alpha|\Omega_A,c,\lambda) = 1, \quad \sum_{\beta} P(\beta|\Omega_B,c,\lambda) = 1. \tag{10}$$

By combining (8) and (10) it is straightforward to show that the joint probability $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ satisfies the standard probability sum rule (B1). Again, using (10) and (11) the general relationships (B3) and (B2) between the joint and single measurement probabilities occur.

The overall probability $P(\alpha|\Omega_A,c)$ that measurement of the *observable* Ω_A of subsystem A results in *outcome* α when the *preparation* process is c irrespective of the outcome for measurement of the *observable* Ω_B of subsystem B is obtained by summing $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ over β [see (B3)], so it is given by the sum over the possible values λ of the hidden variables of the $P(\alpha|\Omega_A,c,\lambda)$ times the preparation probability $P(\lambda|c)$. A similar expression applies for $P(\beta|\Omega_B,c)$. Thus using (9) and (10)

$$\begin{aligned} P(\alpha|\Omega_A,c) &= \sum_{\lambda} P(\alpha|\Omega_A,c,\lambda)\,P(\lambda|c), \\ P(\beta|\Omega_B,c) &= \sum_{\lambda} P(\beta|\Omega_B,c,\lambda)\,P(\lambda|c). \end{aligned} \tag{11}$$

Under the condition of Bell locality, the results (11) show that in a LHVT the measurement probability for an observable Ω_A of subsystem A is independent of the results for measuring an observable Ω_B of subsystem B, and do not even depend on which observable Ω_B is being measured. The same applies if the subsystems are reversed. This important result for LHVT is called the *no-signaling theorem* and shows that a choice of observable to be measured in one subsystem cannot affect the result of measurements in the other subsystem.

The *conditional probabilities* are given by the general expressions (B5) that apply for both quantum and LHVT cases.

We can use (9) to obtain an expression for the *mean value* of the *joint measurement* of observables Ω_A and Ω_B when the preparation process is c. This will be given by

$$\langle \Omega_A \otimes \Omega_B \rangle = \sum_{\alpha,\beta} \alpha\, \beta\, P(\alpha, \beta|\Omega_A, \Omega_B, c) = \sum_{\lambda} \langle \Omega_A(c,\lambda) \rangle \langle \Omega_B(c,\lambda) \rangle P(\lambda|c), \quad (12)$$

where $\langle \Omega_A(c,\lambda) \rangle \equiv \langle \Omega_A(\lambda) \rangle$ is the expectation value of observable Ω_A when the preparation process c leads to hidden variables λ, with $\langle \Omega_B(c,\lambda) \rangle \equiv \langle \Omega_B(\lambda) \rangle$ the corresponding expectation value for observable Ω_B. These are given by

$$\begin{aligned} \langle \Omega_A(c,\lambda) \rangle &= \sum_{\alpha} \alpha\, P(\alpha|\Omega_A, c, \lambda), \\ \langle \Omega_B(c,\lambda) \rangle &= \sum_{\beta} \beta\, P(\beta|\Omega_B, c, \lambda). \end{aligned} \quad (13)$$

The *mean value* for the measurement of a *single observable* Ω_A is

$$\langle \Omega_A \rangle = \sum_{\alpha} \alpha\, P(\alpha|\Omega_A, c) = \sum_{\lambda} \langle \Omega_A(c,\lambda) \rangle P(\lambda|c), \quad (14)$$

as can be derived from (11) and (13). A similar result applies for $\langle \Omega_B \rangle$.

In a *deterministic* (or nonfuzzy) version of LHVT $\langle \Omega_A(c,\lambda) \rangle = \alpha(c,\lambda)$ and $\langle \Omega_B(c,\lambda) \rangle = \beta(c,\lambda)$, where $\alpha(c,\lambda)$ and $\beta(c,\lambda)$ are *specific* allowed outcomes for measurement of the observables when the preparation process c leads to hidden variables λ. Here the hidden variables λ determine *unique* measurement outcomes $\alpha(c,\lambda)$ and $\beta(c,\lambda)$. In the deterministic case

$$\langle \Omega_A \otimes \Omega_B \rangle = \sum_{\lambda} \alpha(c,\lambda)\, \beta(c,\lambda)\, P(\lambda|c), \quad (15)$$

which is a form originally used for $\langle \Omega_A \otimes \Omega_B \rangle$ by Bell (see Ref. [4]). Thus in a nonfuzzy version of LHVT the hidden variables *uniquely* specify the measurement outcomes, and it is only because the hidden variables are *not known* that they must be averaged over.

C. Links between quantum and local hidden variable theory

In accordance with Einstein's basic idea that quantum theory predictions for $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ and $P(\alpha|\Omega_A, c)$, $P(\beta|\Omega_B, c)$ are *correct*, but can be *interpreted* in terms of an underlying *reality* represented by a hidden variable theory, it follows that the *same* joint probability in (9) can *also* be determined from the quantum theory expression (1). Similarly for the single measurement probabilities $P(\alpha|\Omega_A, c)$, $P(\beta|\Omega_B, c)$. Note that this *assumes* that the particular quantum state for the composite system *can* be interpreted via local hidden variable theory, which by definition excludes the *Bell nonlocal* states. As we have already noted, there are *actual* Bell nonlocal states where the quantum results are *not* accountable via LHVT—either *theoretically* or *experimentally*. So it is only when we are considering *Bell local* states that these interrelationships can be applied.

As indicated in Sec. I, a key issue is how to interrelate the Hermitian operators that describe the observables in quantum theory to the c-number variables describing the same observables in LHVT, in order that valid comparisons between the predictions of quantum and LHVT can be made.The approach that will be used is to express all the quantum theory observables of interest in terms of Hermitian operators associated with observables (such as position and momentum) that have a classical counterpart, and then choose the equivalent LHVT observables to have the same form as those in quantum theory, except that the Hermitian operators will be replaced by c-number variables. As indicated in Sec. I it will be necessary to introduce auxiliary observables whose c-number versions have separate probability distributions. The procedure will be discussed in more detail in Sec. IV.

For *Bell local* states, equating the LHVT (11) and quantum theory (3) expressions for the single measurement *probability* $P(\alpha|\Omega_A, c)$ we obtain a LHVT–quantum theory relationship

$$\begin{aligned} P(\alpha|\Omega_A, c) &= \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda)\, P(\lambda|c), && \text{LHVT} \\ &= \mathrm{Tr}\big((\widehat{\Pi}^A_\alpha \otimes \widehat{1}^B)\widehat{\rho}\big). && \text{QT} \end{aligned} \quad (16)$$

As $\mathrm{Tr}((\widehat{\Pi}^A_\alpha \otimes \widehat{1}^B)\widehat{\rho}) = \mathrm{Tr}_A(\widehat{\Pi}^A_\alpha\, \widehat{\rho}^A)$ this shows that the *hidden variable theory probability* $P(\alpha|\Omega_A, c, \lambda)$ associated with single subsystem A measurements and the *reduced density operator* $\widehat{\rho}^A$ for subsystem A are interrelated. A similar result applies for $P(\beta|\Omega_B, c)$. However, this relationship does *not* mean that $P(\alpha|\Omega_A, c, \lambda)$ can always be *determined* from a subsystem density operator which is *not* dependent on the overall quantum state $\widehat{\rho}$ describing *both* subsystems together—in general the reduced density operator for each subsystem is determined from the *full* density operator $\widehat{\rho}$. However, when there is a LHS, the reduced density operator $\widehat{\rho}^A$ may be replaced by the form $\widehat{\rho}^A(c,\lambda)$—which is determined specifically for subsystem A for preparation process c via the hidden variables λ.

Similar considerations apply for *Bell local* states to the joint measurement *probability* $P(\alpha, \beta|\Omega_A, \Omega_B, c)$. We have a second LHVT–quantum theory relationship:

$$\begin{aligned} P(\alpha, \beta|\Omega_A, \Omega_B, c) &= \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda)\, P(\beta|\Omega_B, c, \lambda) \\ &\quad \times P(\lambda|c), \quad \text{LHVT} \\ &= \mathrm{Tr}\big((\widehat{\Pi}^A_\alpha \otimes \widehat{\Pi}^B_\beta)\widehat{\rho}\big). \quad \text{QT} \end{aligned} \quad (17)$$

Also, for *Bell local* states we can interrelate the quantum and LHVT *mean values* of the *joint* measurement of observables Ω_A and Ω_B when the preparation process is c. Using (4) and (12) we have

$$\begin{aligned} \langle \Omega_A \otimes \Omega_B \rangle &= \sum_{\lambda} \langle \Omega_A(c,\lambda) \rangle \langle \Omega_B(c,\lambda) \rangle P(\lambda|c), && \text{LHVT} \\ &= \mathrm{Tr}(\widehat{\Omega}_A \otimes \widehat{\Omega}_B)\widehat{\rho} = \langle \widehat{\Omega}_A \otimes \widehat{\Omega}_B \rangle && \text{QT} \end{aligned} \quad (18)$$

in cases where the LHVT can be applied.

In the case of *mean values* for a *single* observable, we have similarly

$$\begin{aligned} \langle \Omega_A \rangle = \langle \Omega_A \otimes 1_B \rangle &= \sum_{\lambda} \langle \Omega_A(c,\lambda) \rangle P(\lambda|c), && \text{LHVT} \\ &= \mathrm{Tr}[(\widehat{\Omega}_A \otimes \widehat{1}_B)\widehat{\rho}] = \langle \widehat{\Omega}_A \otimes \widehat{1}_B \rangle = \langle \widehat{\Omega}_A \rangle && \text{QT} \end{aligned} \quad (19)$$

for *Bell local* states. A similar result applies for $\langle\Omega_B\rangle$. These results are all useful for *interconverting* LHVT and quantum theory expressions, for the Bell local states.

The above results assume that there is a well-defined *relationship* for the c-numbers that represent the observables Ω_A, Ω_B in LHVT and the Hermitian operators $\widehat{\Omega}_A$, $\widehat{\Omega}_B$ that represent the *same* observables in quantum theory. It is also required that the LHVT involves the same measurement *outcomes* α, β apply as for quantum theory. Other constraints on the LHVT probability distributions would need to be imposed if the LHVT is required to be consistent with quantum theory features such as the *Heisenberg uncertainty principle* for observables with noncommuting quantum operators. This issue is not addressed here.

As previously emphasized, a key point is that because LHVT *underlies* quantum theory, *any* result we establish mean values, variances of observables Ω_A, Ω_B using LHVT for a quantum state that is *also* Bell local, can immediately be expressed in terms of the equivalent Hermitian *operators* observables $\widehat{\Omega}_A$, $\widehat{\Omega}_B$ that describe the same observables, together with the quantum *density operator* $\widehat{\rho}$ that specifies the *same* state instead of the set of LHVT *probabilities* $P(\alpha|\Omega_A,c,\lambda)$, $P(\beta|\Omega_B,c,\lambda)$, and $P(\lambda|c)$. Except in the case of a LHS there are no quantum expressions for quantities such as $P(\alpha|\Omega_A,c,\lambda)$, $\langle\Omega_A(c,\lambda)\rangle$, so no attempt will be made to replace these by quantum expressions. Also, both the Bell inequalities and the tests for EPR steering involve only mean values of various observables, a primary emphasis will be on the two expressions (19) and (18) involving mean values of either single subsystem observables or pairs of such observables.

We will also need to consider the mean values for observables which in quantum theory are given by the *sum* of *products* of subsystem Hermitian operators, where the operators for each subsystem do not necessarily commute—$[\widehat{\Omega}_{A1},\widehat{\Omega}_{A2}]\neq 0$, etc. The links between quantum theory and LHVT for these cases are set out in Appendix C.

III. CATEGORIES OF QUANTUM STATES FOR BIPARTITE SYSTEMS

A. Two hierarchies of bipartite quantum states

As indicated in Sec. I there are various ways the quantum states for bipartite systems can be *categorized*, and quantum states falling into a particular category in one scheme *may not* all end up in the same category in a different scheme. Jones *et al.* [7] (as elaborated by Cavalcanti *et al.* [8]), established a hierarchy of *bipartite quantum states* can be established based on *LHVT models* for the *joint probability* $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ for measurement of *any* pair of subsystem *observables* Ω_A and Ω_B to obtain *any* of their possible *outcomes* α and β when the *preparation* process is c. However, before considering this hierarchy we first identify a *classification* based purely on *quantum state models*.

B. Separable and entangled states

The quantum states for bipartite composite systems may be divided into *two classes*—the *separable* and the *entangled* states. We will refer to this scheme as the *quantum theory classification scheme* (QTCS).

The *separable* states are those whose preparation is described by the density operator

$$\widehat{\rho}_{\text{sep}}=\sum_R P_R\,\widehat{\rho}_R^A\otimes\widehat{\rho}_R^B, \tag{20}$$

where $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ are *possible* quantum states for subsystems A and B, respectively, and P_R is the probability that this *particular pair* of subsystem states is prepared. Each distinct pair is listed by R. This follows the preparation process for separable states described by Werner [12]. Such quantum states are of the same form as what Werner [12] referred to as *uncorrelated states*, but which nowadays would be referred to as separable or *nonentangled* states. The *entangled* states are simply the quantum states that are *not* separable. A detailed discussion of the significance of separable and entangled states, and tests for distinguishing these is given in many articles and textbooks (see, for example, Refs. [2,3]). Clearly for each choice of subsystems a given quantum state is *either* separable *or* entangled—it cannot be both.

For the present we note that *if* the quantum state is *separable*, then from (1) and (20) the joint probability $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ is given by

$$P(\alpha,\beta|\Omega_A,\Omega_B,c)=\sum_R P_R\,\mathrm{Tr}_A\big(\widehat{\Pi}_\alpha^A\,\widehat{\rho}_R^A\big)\,\mathrm{Tr}_B\big(\widehat{\Pi}_\beta^B\,\widehat{\rho}_R^B\big), \tag{21}$$

$$=\sum_R P_R\,P(\alpha|\Omega_A,c(A,R))\,P(\beta|\Omega_B,c(B,R)), \tag{22}$$

where

$$\begin{aligned}P(\alpha|\Omega_A,c(A,R))&=\mathrm{Tr}_A\big(\widehat{\Pi}_\alpha^A\,\widehat{\rho}_R^A\big),\\ P(\beta|\Omega_B,c(B,R))&=\mathrm{Tr}_B\big(\widehat{\Pi}_\beta^B\,\widehat{\rho}_R^B\big)\end{aligned} \tag{23}$$

are the *subsystem probabilities* for outcomes α, β for measurements of observables Ω_A, Ω_B when the subsystem preparations specify density operators as $c(A,R)\rightarrow\widehat{\rho}_R^A$, $c(B,R)\rightarrow\widehat{\rho}_R^B$.

Alternatively, *if* the joint probability is given by (21) for *all* observables and outcomes then we can show that $P(\alpha,\beta|\Omega_A,\Omega_B,c)=\mathrm{Tr}((\widehat{\Pi}_\alpha^A\otimes\widehat{\Pi}_\beta^B)\widehat{\rho})$, where $\widehat{\rho}=\sum_R P_R\,\widehat{\rho}_R^A\otimes\widehat{\rho}_R^B$—so the state is separable. Thus the density operator definition and the joint probability expression for a separable state are *equivalent*.

C. Bell local and nonlocal states

Based on LHVT the quantum states for bipartite composite systems may *also* be *differently* divided into *two other classes*—the *Bell local* and the *Bell-nonlocal* states. We will refer to this scheme as the *local hidden variable theory classification scheme* (LHVTCS). As we will see, there is *no simple* relationship between the entangled states on the one hand and the Bell nonlocal states on the other, (nor between the separable states on the one hand and the Bell local states on the other). The *Bell local* states are those for which the joint probability $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ is given by the LHVT expression (9) *as well as* the quantum theory expression (1). In contrast, the *Bell nonlocal* states are those for which there is *no* LHVT expression (9) for the joint probability—this is given *only* by the quantum theory expression (1).

Before looking at *further classes* of quantum states defined in terms of LHVT we first present an important result, namely, that *all separable* states are *Bell local*. The *formal similarity* between the hidden variable theory expression for the joint probability (9) and the quantum expression (22) for a *separable* state is noticeable. We can then identify the probabilistic choice R for the preparation of the *particular pair* of subsystem states $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ with a *particular choice* of hidden variables λ, thus $R \rightarrow \lambda$. The $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ thus specify *local hidden states*. Then the probability P_R for this particular pair of subsystem states $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ can be identified with the hidden variable probability $P(\lambda|c)$, thus $P_R \rightarrow P(\lambda|c)$. Next, the probabilities $P(\alpha|\Omega_A, c(A,R))$ and $P(\beta|\Omega_B, c(B,R))$ for the single subsystem probabilities can be identified with the hidden variable probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$, thus $P(\alpha|\Omega_A, c(A,R)) \rightarrow P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda) \rightarrow P(\beta|\Omega_B, c, \lambda)$. With these identifications the joint probability $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ for a separable state (22) *is* of the general form for the joint probability $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ for a Bell local state (9). Hence the separable states are Bell local.

Thus, for the quantum *separable* states the joint probability can be written as

$$P(\alpha, \beta|\Omega_A, \Omega_B, c) = \sum_{\lambda} P_Q(\alpha|\Omega_A, c, \lambda)\, P_Q(\beta|\Omega_B, c, \lambda) P(\lambda|c) \quad (24)$$

where the single probabilities are given by *quantum theory* expressions

$$\begin{aligned} P(\alpha|\Omega_A, c, \lambda) &= \mathrm{Tr}_A\big(\widehat{\Pi}_\alpha^A\, \widehat{\rho}_R^A\big) = P_Q(\alpha|\Omega_A, c, \lambda), \\ P(\beta|\Omega_B, c, \lambda) &= \mathrm{Tr}_B\big(\widehat{\Pi}_\beta^B\, \widehat{\rho}_R^B\big) = P_Q(\beta|\Omega_B, c, \lambda), \end{aligned} \quad (25)$$

where the subscript Q indicates that a quantum theory expression applies.

It therefore follows that *all* Bell *nonlocal* states are *quantum entangled*. After all, if the quantum state is Bell nonlocal and is also separable, then the separable state expression (22) applies for the joint measurement probability, which being of the required form for LHVT leads to the contradictory result that the state was Bell local. Thus, *all* quantum separable states are *Bell local* and *all* Bell nonlocal states are quantum *entangled*. Note, however, that the converses are *not* true. As we will see, *some* Bell local states are *not* quantum separable, that is they are quantum entangled. Similarly, *some* quantum entangled states are *not* Bell nonlocal, that is, they are Bell local. This last result was established by Werner [12].

D. Categories of Bell local states

This situation for separable states suggests that the *Bell local states* for *bipartite* systems may be divided up into *three* classes depending on the *number* of single subsystem probabilities that are *definitely* described by quantum expressions involving the *density operator* $\widehat{\rho}^C(c, \lambda)$ for a *LHS* and a *projector* $\widehat{\Pi}_\omega^C$ associated with measurement outcome ω for observable $\widehat{\Omega}_C$. For bipartite systems there are three possibilities: first, *Category 1* states where *both* $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$ are given by quantum expressions as in (25); second, *Category 2* states where only *one* is given by a quantum expression; and third, *Category 3* states where *neither* is given by a quantum expression. The three classes or categories are mutually exclusive—a given Bell local state can only be in *one* of the three classes. We now introduce a *different* notation in which [as in Eq. (25)] the *presence* of the subscript Q on a subsystem LHV probability indicates that it *can* be obtained from a quantum expression involving a subsystem density operator for a LHS, and the *absence* of the subscript Q indicates that it is *not* determined from a quantum expression. Note that our notation *differs* from that in Refs. [6–8] where the $P(\alpha|\Omega_A, c, \lambda)$ could be *either* $P(\alpha|\Omega_A, c, \lambda)$ (nonquantum) *or* $P_Q(\alpha|\Omega_A, c, \lambda)$ (quantum) in our notation. Hence in the present notation the joint probabilities for the *Bell local* states in *Categories 1, 2, and 3* are given by

$$P(\alpha, \beta|\Omega_A, \Omega_B, c) = \sum_{\lambda} P_Q(\alpha|\Omega_A, c, \lambda)\, P_Q(\beta|\Omega_B, c, \lambda) \times P(\lambda|c), \qquad \text{Category 1} \quad (26)$$

$$P(\alpha, \beta|\Omega_A, \Omega_B, c) = \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda)\, P_Q(\beta|\Omega_B, c, \lambda) \times P(\lambda|c), \qquad \text{Category 2} \quad (27)$$

$$P(\alpha, \beta|\Omega_A, \Omega_B, c) = \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda)\, P(\beta|\Omega_B, c, \lambda) \times P(\lambda|c). \qquad \text{Category 3} \quad (28)$$

When a quantum expression applies:

$$\begin{aligned} P_Q(\alpha|\Omega_A, c, \lambda) &= \mathrm{Tr}_A\big(\widehat{\Pi}_\alpha^A\, \widehat{\rho}^A(c, \lambda)\big), \\ P_Q(\beta|\Omega_B, c, \lambda) &= \mathrm{Tr}_B\big(\widehat{\Pi}_\beta^B\, \widehat{\rho}^B(c, \lambda)\big), \end{aligned} \quad (29)$$

where $\widehat{\rho}^A(c, \lambda)$ and $\widehat{\rho}^B(c, \lambda)$ are the subsystem density operators for the LHSs associated with hidden variables λ for preparation c. By *convention* for Category 2 states we choose B to be the subsystem where the single probability is given by a quantum expression.

We also list as *Category 4* states those for which the joint probability is *not* given by *any* of Eqs. (26), (27), and (28):

$$P(\alpha, \beta|\Omega_A, \Omega_B, c) \neq \text{Eqs. (26), (27), or (28).} \qquad \text{Category 4} \quad (30)$$

For these states the joint probability is *only* given by the quantum theory expression (1). The Category 4 states are of course the *Bell nonlocal* states, and such states *do* occur. If Einstein's realist approach applied there would be *no* Category 4 states.

To avoid confusion we note that Wiseman *et al.* [6] also introduced the term *local hidden state model* to refer to the situation when *at least one* subsystem is associated with a LHS. Thus the *LHS model* applies to Category 1 and Category 2 states, but not to Category 3 and Category 4 states.

Clearly, all separable states are Category 1 states, and all Category 1 states are separable. The *Category 1* states may also be just referred to as *separable* states. However, *Category 2*, *Category 3* and *Category 4* states must be quantum *entangled* states. The four different categories of bipartite states have differing features in regard to *entanglement* based on their distinction via the number of subsystems associated with a *local hidden states*.

The feature of *EPR steering* of subsystem B from subsystem A is fully discussed in Refs. [6–8] and requires there must be *no* LHS $\widehat{\rho}^{B}(c,\lambda)$ for subsystem B. For such states the subsystem B said to be *nonsteerable* from subsystem A. For completeness, a brief presentation of the physical argument involved based on a consideration of states that are conditional on the outcomes of measurements on subsystem A, is set out in Appendix F. Thus EPR steering requires the *failure* of the *LHS model*. Hence Category 1 and Category 2 states are *nonsteerable*, whereas Category 3 and Category 4 states are *steerable* since no LHS for subsystem B is involved. The Category 3 states, which are Bell local, entangled, non-LHS, and steerable are sometimes referred to as *EPR entangled* states. Thus, based on their distinction via the number of subsystems associated with a LHS, the four different categories of bipartite states also have differing features in regard to *EPR steering*.

As we have now seen, the Bell local states for *bipartite* systems can be divided up into three *nonoverlapping* subsets, each of which has different features for the subsystem LHV probabilities $P(\alpha|\Omega_A,c,\lambda)$ and $P(\beta|\Omega_B,c,\lambda)$. This distinctiveness between the subsets is of particular convenience when we consider tests for various categories of states. However, it should again be emphasized that other researchers (Refs. [6,7] and [8]) have used a *hierarchy* of nondisjoint subsets. This is because in certain of their definitions the subsystem probabilities can be either given by quantum or nonquantum expressions. In their scheme the subsets overlap, with each set being a subset of a larger set. In their scheme Category 1 states (the separable states) would be a subset of a set (the LHSs) consisting of Category 1 and Category 2 states, where at least one subsystem is in a LHS. In their scheme the Category 1 and Category 2 states would be combined and be a subset of a combined set (the Bell local states) consisting of Category 1, Category 2, and Category 3 states. Thus the present scheme and that in Refs. [6,7] and [8] are *not* the same though they are *related*, and this needs to be taken into account when discussing tests. The *overall scheme* used *here* is shown in Fig. 1, where the features for all the different sets of states for bipartite composite systems are set out.

The mixed states introduced by Werner [12] provide examples of the three categories of Bell local states and of the Bell nonlocal states. These are certain $U\otimes U$ invariant states $[(\widehat{U}\otimes\widehat{U})\,\widehat{\rho}_W\,(\widehat{U}^{\dagger}\otimes\widehat{U}^{\dagger})=\widehat{\rho}_W$, where $\widehat{U}$ is any *unitary* operator] for two d-dimensional subsystems. Depending on the parameter η (or ϕ) the Werner states [see Eq, (E1)] may be separable or entangled. They may also be Bell local and in one of the three categories described above, or they may be Bell nonlocal. For completeness the Werner states are described in Appendix E. The GHZ (or maximally entangled) pure state for two subsystems, each consisting of a spin 1/2 particle considered by He *et al.* [13], and given by $(|\frac{1}{2},+\frac{1}{2}\rangle_A|\frac{1}{2},+\frac{1}{2}\rangle_B+|\frac{1}{2},-\frac{1}{2}\rangle_A|\frac{1}{2}-\frac{1}{2}\rangle_B)/\sqrt{2}$ is an example of a Category 3 state, since it is entangled and steerable, but is still Bell-local. As mentioned previously, the singlet state [27] for the same system—given by $(|\frac{1}{2},+\frac{1}{2}\rangle_A|\frac{1}{2},+\frac{1}{2}\rangle_B-|\frac{1}{2},-\frac{1}{2}\rangle_A|\frac{1}{2}-\frac{1}{2}\rangle_B)/\sqrt{2}$—is an example of a Category 4 state, since it is entangled, steerable, and is Bell nonlocal as it violates a Bell inequality.

QUANTUM THEORY	LOCAL HIDDEN VARIABLE THEORY	LHVT CATEGORY	QUANTUM THEORY FEATURES	LHVT FEATURES
Separable states	Bell local states	Category 1	Quantum separable	LHS state Non steer Bell local
Quantum entangled states		Category 2	Quantum entangled	LHS state Non steer Bell local
		Category 3	Quantum entangled	Steerable Bell local
	Bell non-local states	Category 4	Quantum entangled	Steerable Bell non-local

FIG. 1. The quantum theory and the local hidden variable theory classification schemes (QTCS and LHVCS). The two categories of quantum states in the QTCS are shown in the left column, and the two basic categories of quantum states in the LHVCS are shown in the second left column. The four more detailed categories of quantum states in the LHVCS are shown in the third left column, while the right two columns lists the features of the four categories of LHVCS states in both the QTCS and LHVCS schemes.

IV. TESTS FOR EPR STEERING IN BIPARTITE SYSTEMS

A. General considerations

In a number of papers (see the review papers [2,3] and references therein) various tests for quantum *entanglement* have been formulated, recently in the particular context of bipartite systems of identical massive bosons [1]. The focus was on the situation of single-mode subsystems. These include spin and two-mode quadrature squeezing, Bloch vector and correlation tests. An important issue then is: Are these tests also valid for detecting *EPR steering* or do some of them fail? As for the entanglement tests, for the EPR steering tests we also focus on *single-mode* subsystems. Of course any test that detects EPR steering must of necessity also detect entanglement, but a test that demonstrates entanglement does not necessarily demonstrate EPR steering. In this situation we are looking for conditions where there is no LHS for subsystem B—or in other words, the quantum state does *not* have a joint measurement probability as in Eqs. (26) and (27) for Category 1 or Category 2 states. Thus EPR steering requires the failure of the LHS model. As the tests for *quantum entanglement* previously obtained have already found the conditions under which Category 1 probabilities fail, we *then* know that the quantum state must be in Category 2, Category 3, or Category 4. If we can then show that it is *not* in Category 2 because the joint measurement probability (27) also fails, then the state *must* be in Category 3 or Category 4—in other words it is an *EPR steerable* state. We would then have found a *test* for *EPR steering*. Note that for the Category 2 states the subsystem A probabilities $P(\alpha|\Omega_A, c, \lambda)$ in LHVT are *not* given by a quantum expression involving a *subsystem* density operator. *This feature* must be taken into account when considering the tests for EPR steering. However, the issue of how to treat mean values and variances in the context of LHVT in general requires some consideration, so we have set this out in Appendix C.

Note, however, that a test that demonstrates EPR steering only shows that the quantum state is either Category 3 or Category 4, both of which are entangled states. To demonstrate *Bell nonlocality* (Category 4 states) will require different tests—notably those involving violations of a *Bell inequality*. This will be the subject of a later paper. As has been emphasized in Sec. I, showing that a Bell inequality is violated demonstrates that the state cannot be in Category 1, 2, or 3, so it must be a Bell nonlocal state (Category 4). However, we emphasize again the point that the tests presented here show what category (or categories) the quantum state *cannot* belong to—which does *not always* determine what category of quantum state *must* apply. The tests are those of *sufficiency* not *necessity*.

In the present paper, as in previous work in Refs. [1–3], we focus on tests for bipartite systems involving *identical massive bosons*. Consequently, when quantum states either for the overall system or for a subsystem are involved these must comply with the *symmetrization* principle and *super-selection rules* involving the total boson number for either the overall system or for the subsystem. In particular, for Category 2 states (as well as Category 1 states) the LHS $\widehat{\rho}^B(c, \lambda)$ for the subsystem B that is treated quantum mechanically must have zero coherences between Fock states with differing subsystem boson number N_B. The LHS must be a possible quantum state for subsystem B. The issue of SSRs is discussed fully in Ref. [2].

Also, as in these papers both the overall system and the two subsystems will be specified in terms of *modes* (or single-particle states that the particles may occupy) based on a second quantization treatment, rather than in terms of labeled identical *particles*, as might be thought appropriate in a first quantization method. Cases with differing *numbers* of particles are just different *states* of the (multi-) modal system, not different systems, as in first quantization.

In addition, since the mean values of various observables are involved in the tests for showing the state is not Category 2, we can use Eqs. (19) and (18) for overall system mean values to replace LHVT theory expressions by quantum theory expressions at suitable stages in the derivations—both when a subsystem B LHS $\widehat{\rho}^B(c, \lambda)$ occurs or when we wish to evaluate the mean value of a subsystem A observable Ω_A allowing for all values of the hidden variables λ. However, there will be situations for Category 2 states where we need to consider the mean value of a subsystem A observable Ω_A when the hidden variables have particular values. In this case some general properties of classical probabilities $P(\alpha|\Omega_A, c, \lambda)$ are useful. These are not dependent on $P(\alpha|\Omega_A, c, \lambda)$ being obtained from a hidden state density operator $\widehat{\rho}^A(c, \lambda)$. One is that the mean of the square of a real observable is never less than the square of the mean for the observable, that is,

$$\langle \Omega_A^2(c, \lambda)\rangle \geqslant [\langle \Omega_A(c, \lambda)\rangle]^2. \tag{31}$$

Another is a Cauchy inequality

$$\sum_\lambda C(\lambda)\, P(\lambda|c) \geqslant \left[\sum_\lambda \sqrt{C(\lambda)}\, P(\lambda|c)\right]^2 \tag{32}$$

for $C(\lambda) \geqslant 0$, such as the case $C(\lambda) = \langle \Omega_A^2(c, \lambda)\rangle$. The proof of the first is elementary, the second is proved in Ref. [2]. These results are used only to derive correlation tests (see Appendix I).

Finally, since LHVT deals with physical quantities that are *classical* observables we need to express various non-Hermitian quantum mechanical operators that we need to consider—such as mode annihilation and creation operators—in terms of quantum operators that are Hermitian. Any non-Hermitian operator $\widehat{\Omega}$ can always be expressed in terms of Hermitian operators $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ as $\widehat{\Omega} = \widehat{\Omega}_1 + i\,\widehat{\Omega}_2$ and the latter operators would be equivalent to classical observables Ω_1 and Ω_2, so the corresponding classical observable will be $\Omega = \Omega_1 + i\,\Omega_2$. The mean value $\langle\widehat{\Omega}\rangle$ will then be equal to $\langle\widehat{\Omega}_1\rangle + i\langle\widehat{\Omega}_2\rangle$. Note that two independent sets of measurements for the generally incompatible $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ would be needed to separately determine $\langle\widehat{\Omega}_1\rangle$ and $\langle\widehat{\Omega}_2\rangle$. For the corresponding classical observable we take $\langle\Omega\rangle = \langle\Omega_1\rangle + i\langle\Omega_2\rangle$—see Eq. (C28) in Appendix C. The bosonic annihilation and creation operators for each of the single-mode subsystems are not Hermitian, so we replace these by pairs of *quadrature operators* $\widehat{x}$, $\widehat{p}$, which are then associated with classical *quadrature observables* x, p when LHVT is being considered. As we will see, we also need new *auxiliary* Hermitian operators $\widehat{U}, \widehat{V}$ as well, which are sums of products of quadrature operators and these will also be associated with classical observables U, V in the LHVT. All the physical observables that we need to consider have quantum operators that can be written as linear combinations of products $\widehat{\Omega}_A \otimes \widehat{\Omega}_B$, where both $\widehat{\Omega}_A$ and $\widehat{\Omega}_B$ are Hermitian—including cases where $\widehat{\Omega}_A = \widehat{1}_A$ or $\widehat{\Omega}_B = \widehat{1}_B$. Such products can then be replaced by $\Omega_A \otimes \Omega_B$, where Ω_A and Ω_B are the corresponding classical observables. Using this procedure both quantum and hidden variable theory expressions can be used for the joint measurement probabilities and mean values.

B. Spin and quadrature tests for EPR steering

We now obtain a number of inequalities for spin and quadrature observables that apply for Category 2 (and Category 1) states and apply these to obtain tests for EPR steering. First, we consider whether tests that have been shown to be sufficient to demonstrate quantum entanglement (violation of Category 1) (see Ref. [3] for details) are also valid for demonstrating EPR steering. Obviously a test that demonstrates EPR steering must also demonstrate quantum entanglement, but a test that demonstrates entanglement does not necessarily demonstrate EPR steering. We first consider the Bloch vector tests, then spin squeezing tests for S_z and for the other spin components, followed by planar spin variance tests (such as the Hillery-Zubairy test) which involve the sum of the variances for S_x and S_y, and finally two-mode quadrature squeezing tests. Of these possible tests, the Bloch vector test, spin squeezing in any spin component, the Hillery-Zubairy spin variance test and squeezing in any two-mode quadrature are valid for demonstrating EPR steering. We also consider a generalized version of the Hillery- Zubairy spin variance test, which also shows that EPR steering occurs. Finally, we consider for completeness weak and strong correlation tests in Appendix I, though these are equivalent to certain of the tests involving spin operators already set out in this section.

C. Quadrature amplitudes

The non-Hermitian quantum mode annihilation or creation operators can be replaced by their Hermitian components,which are the *quadrature operators*. In quantum theory these are given by

$$\widehat{x}_A = \frac{1}{\sqrt{2}}(\widehat{a}+\widehat{a}^\dagger), \quad \widehat{p}_A = \frac{1}{\sqrt{2}i}(\widehat{a}-\widehat{a}^\dagger),$$
$$\widehat{x}_B = \frac{1}{\sqrt{2}}(\widehat{b}+\widehat{b}^\dagger), \quad \widehat{p}_B = \frac{1}{\sqrt{2}i}(\widehat{b}-\widehat{b}^\dagger), \tag{33}$$

which have the same commutation rules as the position and momentum operators for distinguishable particles in units where $\hbar = 1$. Thus $[\widehat{x}_A, \widehat{p}_A] = [\widehat{x}_B, \widehat{p}_B] = i$ as for cases where A, B were distinguishable particles. It is then reasonable to assume that there are equivalent classical observables x_A, p_A, x_B, p_B and that their measurement outcomes would be real numbers, and further more for subsystems not being treated quantum mechanically (such as subsystem A in the context of Category 2 states) these outcomes can *actually* be measured in *experiment* and probabilities and mean values such as $P(\alpha|\Omega_A, c, \lambda)$ and $\langle\Omega_A(\lambda)\rangle$ can be assigned as in a hidden variable treatment of subsystem A. However, in considering Category 2 states the probabilities and mean values such as $P(\beta|\Omega_B, c, \lambda)$ and $\langle\Omega_B(\lambda)\rangle$ for the subsystem B are also given by quantum expressions involving subsystem density operators $\widehat{\rho}^B(\lambda)$.

We can write the mode annihilation and creation operators in terms of the quadrature operators as $\widehat{a} = (\widehat{x}_A + i\widehat{p}_A)/\sqrt{2}$, $\widehat{a}^\dagger = (\widehat{x}_A - i\widehat{p}_A)/\sqrt{2}$, $\widehat{b} = (\widehat{x}_B + i\widehat{p}_B)/\sqrt{2}$, $\widehat{b}^\dagger = (\widehat{x}_B - i\widehat{p}_B)/\sqrt{2}$ and then show that important observables can be expressed in terms of the quadrature operators. In the case of the *spin operators* [defined as $\widehat{S}_x = (\widehat{b}^\dagger\widehat{a} + \widehat{a}^\dagger\widehat{b})/2$, $\widehat{S}_y = (\widehat{b}^\dagger\widehat{a} - \widehat{a}^\dagger\widehat{b})/2i$, $\widehat{S}_z = (\widehat{b}^\dagger\widehat{b} - \widehat{a}^\dagger\widehat{a})/2$] and the *number operators* [defined as $\widehat{N} = \widehat{N}_A + \widehat{N}_B$ with $\widehat{N}_A = \widehat{a}^\dagger\widehat{a}$, $\widehat{N}_B = \widehat{b}^\dagger\widehat{b}$ being the separate mode number operators—note that $\widehat{S}_x^2 + \widehat{S}_y^2 + \widehat{S}_z^2 = \frac{\widehat{N}}{2}(\frac{\widehat{N}}{2} + 1)$], all these quantities can be expressed in terms of the quadrature operators as follows:

$$\widehat{S}_x = \tfrac{1}{2}(\widehat{x}_A\widehat{x}_B + \widehat{p}_A\widehat{p}_B), \quad \widehat{S}_y = \tfrac{1}{2}(\widehat{p}_A\widehat{x}_B - \widehat{x}_A\widehat{p}_B),$$
$$\widehat{S}_z = \tfrac{1}{4}\big(\widehat{x}_B^2 - \widehat{x}_A^2 + \widehat{p}_B^2 - \widehat{p}_A^2\big) - \tfrac{1}{2}\widehat{V}_B + \tfrac{1}{2}\widehat{V}_A,$$
$$\widehat{N} = \tfrac{1}{2}\big(\widehat{x}_B^2 + \widehat{x}_A^2 + \widehat{p}_B^2 + \widehat{p}_A^2\big) - \widehat{V}_B - \widehat{V}_A, \tag{34}$$

which are all linear combinations of products of two quadrature operators. Here we have introduced the *auxiliary* Hermitian operators

$$\widehat{V}_A = \frac{1}{2i}\big(\widehat{x}_A\widehat{p}_A - \widehat{p}_A\widehat{x}_A\big) = \frac{1}{2}\widehat{1}_A\,,$$
$$\widehat{V}_B = \frac{1}{2i}\big(\widehat{x}_B\widehat{p}_B - \widehat{p}_B\widehat{x}_B\big) = \frac{1}{2}\widehat{1}_B\,, \tag{35}$$

using the commutation rules. These operators could represent observables in quantum theory, albeit rather useless ones since all eigenstates have the same eigenvalue of $1/2$. In terms of the quadrature and auxiliary operators the *mode number* and mode *number difference* operators are

$$\widehat{N}_A = \tfrac{1}{2}\big(\widehat{x}_A^2 + \widehat{p}_A^2\big) - \widehat{V}_A,$$
$$\widehat{N}_B = \tfrac{1}{2}\big(\widehat{x}_B^2 + \widehat{p}_B^2\big) - \widehat{V}_B, \tag{36}$$

$$\widehat{N}_- = \widehat{N}_B - \widehat{N}_A = 2\widehat{S}_z,$$
$$= \tfrac{1}{2}\big(\widehat{x}_B^2 + \widehat{p}_B^2 - \widehat{x}_A^2 - \widehat{p}_A^2\big) - \widehat{V}_B + \widehat{V}_A. \tag{37}$$

As *spin squeezing* was a test for *entanglement* [3], spin squeezing expressions for $\widehat{S}_x^2$, $\widehat{S}_y^2$, and $\widehat{S}_z^2$ will be required. We find that for $\widehat{S}_x^2$ and $\widehat{S}_y^2$,

$$\widehat{S}_x^2 = \tfrac{1}{4}\big(\widehat{x}_A^2\widehat{x}_B^2 + \widehat{p}_A^2\widehat{p}_B^2\big) + \tfrac{1}{2}(\widehat{U}_A\widehat{U}_B - \widehat{V}_A\widehat{V}_B), \tag{38}$$

$$\widehat{S}_y^2 = \tfrac{1}{4}\big(\widehat{p}_A^2\widehat{x}_B^2 + \widehat{x}_A^2\widehat{p}_B^2\big) - \tfrac{1}{2}(\widehat{U}_A\widehat{U}_B + \widehat{V}_A\widehat{V}_B). \tag{39}$$

The spin operators thus involve the quadrature operators for both modes. Here we have introduced two *further* distinct auxiliary Hermitian combinations of the quadrature operators for each mode:

$$\widehat{U}_A = \frac{1}{2}(\widehat{x}_A\widehat{p}_A + \widehat{p}_A\widehat{x}_A) = \frac{1}{2i}[(\widehat{a})^2 - (\widehat{a}^\dagger)^2],$$
$$\widehat{U}_B = \frac{1}{2}(\widehat{x}_B\widehat{p}_B + \widehat{p}_B\widehat{x}_B) = \frac{1}{2i}[(\widehat{b})^2 - (\widehat{b}^\dagger)^2], \tag{40}$$

where using the commutation rules the operators $\widehat{U}_A$ and $\widehat{U}_B$ can also be expressed in terms of mode annihilation and creation operators.

In addition to the spin operators we can also define *two-mode quadrature operators* in terms of the quadrature operators for both modes [3]. These depend on a *phase parameter* θ. There are two sets given by

$$\widehat{X}_\theta(\pm) = \frac{1}{2}(\widehat{a}\,e^{-i\theta} \pm \widehat{b}\,e^{+i\theta} + \widehat{a}^\dagger\,e^{+i\theta} \pm \widehat{b}^\dagger\,e^{-i\theta}),$$
$$\widehat{P}_\theta(\pm) = \frac{1}{2i}(\widehat{a}\,e^{-i\theta} \mp \widehat{b}\,e^{+i\theta} - \widehat{a}^\dagger\,e^{+i\theta} \pm \widehat{b}^\dagger\,e^{-i\theta}). \tag{41}$$

It is easy to see that $\widehat{P}_\theta(\pm) = \widehat{X}_{\theta+\pi/2}(\pm)$ and that $[\widehat{X}_\theta(+), \widehat{P}_\theta(+)] = [\widehat{X}_\theta(-), \widehat{P}_\theta(-)] = i$. The Heisenberg uncertainty principle is given by $\langle\Delta X_\theta^2(\pm)\rangle\langle\Delta P_\theta^2(\pm)\rangle \geqslant 1/4$, and a state is two-mode quadrature *squeezed* if one of $\langle\Delta X_\theta^2(\pm)\rangle$ or $\langle\Delta P_\theta^2(\pm)\rangle$ is less than $1/2$. In Ref. [3] we showed that *two-mode quadrature squeezing* was a sufficiency test for *entanglement*. We can write the two-mode quadrature operators in terms of the single-mode quadrature operators as

$$\widehat{X}_\theta(\pm) = \frac{1}{\sqrt{2}}(\widehat{x}_A\cos\theta + \widehat{p}_A\sin\theta \pm \widehat{x}_B\cos\theta \pm \widehat{p}_B\sin\theta),$$
$$\widehat{P}_\theta(\pm) = \frac{1}{\sqrt{2}}(-\widehat{x}_A\sin\theta + \widehat{p}_A\cos\theta \mp \widehat{x}_B\sin\theta \pm \widehat{p}_B\cos\theta). \tag{42}$$

The square of the two-mode quadrature operators $\widehat{X}_\theta(\pm)$ is given by

$$\begin{aligned}\widehat{X}_\theta(\pm)^2 = {} & \tfrac{1}{2}\{\widehat{x}_A^2\cos^2\theta + \widehat{p}_A^2\sin^2\theta + 2\widehat{U}_A\sin\theta\cos\theta\} \\ & + \tfrac{1}{2}\{\widehat{x}_B^2\cos^2\theta + \widehat{p}_B^2\sin^2\theta + 2\widehat{U}_B\sin\theta\cos\theta\} \\ & \pm\{\widehat{x}_A\widehat{x}_B\cos^2\theta + \widehat{p}_A\widehat{p}_B\sin^2\theta \\ & + \widehat{x}_A\widehat{p}_B\sin\theta\cos\theta + \widehat{p}_A\widehat{x}_B\sin\theta\cos\theta\}. \qquad (43)\end{aligned}$$

The expression for $\widehat{P}_\theta(\pm)^2$ can be obtained using $\widehat{P}_\theta(\pm) = \widehat{X}_{\theta+\pi/2}(\pm)$.

The fundamental quantum Hermitian operators $\widehat{x}_A$, $\widehat{p}_A$, $\widehat{x}_B$, $\widehat{p}_B$ for the two-mode system plus the auxiliary Hermitian operators $\widehat{U}_A$, $\widehat{V}_A$, $\widehat{U}_B$, $\widehat{V}_B$ all correspond to physical quantities that could be measured, with real eigenvalues as the outcomes. Following the general approach described in Sec. I, for local hidden variable theory these quantities correspond to classical observables x_A, p_A, x_B, p_B and U_A, V_A, U_B, V_B, for which single observable hidden variable probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$ apply—from which joint probabilities $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ can be obtained via (9). The physical observables involved in the tests such as the spin operators, their squares and the number operators can all be expressed in terms of the quadrature and auxiliary operators as sums of products of the form $\widehat{\Omega}_A \otimes \widehat{\Omega}_B$. For the local hidden variable theory treatment the corresponding classical observables will be the *same* as the quantum expressions, but now with the quantum Hermitian operators *replaced* by the corresponding classical observable. For the classical spin components S_x, S_y and S_z and the number observable N the expressions in terms of quadrature amplitudes x, p and auxiliary observables U, V are

$$\begin{aligned} S_x &= \tfrac{1}{2}(x_Ax_B + p_Ap_B), \quad S_y = \tfrac{1}{2}(p_Ax_B - x_Ap_B), \\ S_z &= \tfrac{1}{4}\left(x_B^2 - x_A^2 + p_B^2 - p_A^2\right) - \tfrac{1}{2}V_B + \tfrac{1}{2}V_A, \\ N &= \tfrac{1}{2}\left(x_B^2 + x_A^2 + p_B^2 + p_A^2\right) - V_B - V_A. \qquad (44)\end{aligned}$$

The expressions in terms of quadrature amplitudes x, p and auxiliary observables U, V for the subsystem particle numbers and their difference are

$$\begin{aligned} N_A &= \tfrac{1}{2}\left(x_A^2 + p_A^2\right) - V_A, \quad N_B = \tfrac{1}{2}\left(x_B^2 + p_B^2\right) - V_B, \\ N &= N_A + N_B, \\ N_- &= N_B - N_A = 2S_z, \\ &= \tfrac{1}{2}\left(x_B^2 + p_B^2 - x_A^2 - p_A^2\right) - V_B + V_A. \qquad (45)\end{aligned}$$

The two-mode quadrature observables are given by

$$\begin{aligned} X_\theta(\pm) &= \frac{1}{\sqrt{2}}(x_A\cos\theta + p_A\sin\theta \pm x_B\cos\theta \pm p_B\sin\theta), \\ P_\theta(\pm) &= \frac{1}{\sqrt{2}}(-x_A\sin\theta + p_A\cos\theta \mp x_B\sin\theta \pm p_B\cos\theta). \end{aligned} \qquad (46)$$

For completeness we set out expressions for other observables in Appendix D. The reverse process for the replacement of the classical observables x_A, x_B, p_A, p_B by $\widehat{x}_A$, $\widehat{x}_B$, $\widehat{p}_A$, $\widehat{p}_B$ and U_A, U_B, V_A, V_B by $\widehat{U}_A$, $\widehat{U}_B$, $\widehat{V}_A$, $\widehat{V}_B$ requires using (33), (40), and (35) to give the correct quantum Hermitian operators. This requires writing $V_A = (x_Ap_A - p_Ax_A)/2i$ and $U_A = (x_Ap_A + p_Ax_A)/2$, etc., before substituting x_A by $\widehat{x}_A$, p_A by $\widehat{p}_A$ etc., rather than $V_A = 0$ and $U_A = 2x_Ap_A$, etc., but this is not surprising as c-number variables are not mathematically identical to Hermitian operators. Carrying out this replacement in the *classical* spin components S_x, S_y, and S_z and the number observable N also gives the correct *quantum* operators, as also occurs for the squares of these observables as well. Once again we emphasise that we only need single measurement LHVT probabilities $P(\alpha|\Omega_A, c, \lambda)$ with $\Omega_A = x_A$, p_A, U_A or V_A and $P(\beta|\Omega_B, c, \lambda)$ with $\Omega_B = x_B$, p_B, U_B or V_B to treat the classical observables such as S_x, S_y, and S_z and N or $X_\theta(\pm)$, $P_\theta(\pm)$ via hidden variable theory.

The local hidden variable theory for these new observables is defined by measurement probability functions for each subsystem. For subsystem A this will be $P(\alpha_A, \beta_A, \xi_A, \eta_A|x_A, p_A, U_A, V_A, c, \lambda)$ for the measurement outcomes $\alpha_A, \beta_A, \xi_A, \eta_A$ for x_A, p_A, U_A and V_A respectively, with an analogous probability for x_B, p_B, U_B, and V_B. Note that as the measurement outcomes for V_A and V_B are required to be the same as in quantum theory for *any* choice of preparation probability $P(\lambda|c)$, we must have

$$\begin{aligned} & P(\alpha_A, \beta_A, \xi_A, \eta_A|x_A, p_A, U_A, V_A, c, \lambda) \\ & \quad = \delta_{\eta_A,1/2}Q(\alpha_A, \beta_A, \xi_A|x_A, p_A, U_A, c, \lambda), \\ & P(\alpha_B, \beta_B, \xi_B, \eta_B|x_B, p_B, U_B, V_B, c, \lambda) \\ & \quad = \delta_{\eta_B,1/2}Q(\alpha_B, \beta_B, \xi_B|x_B, p_B, U_B, c, \lambda). \qquad (47)\end{aligned}$$

These requirements have implications for the mean values $\langle V_A(\lambda)\rangle$, though only the final mean value $\langle V_A\rangle$ is required for the EPR steering tests.

D. Bloch vector test for EPR steering

1. *Mean values of spin components* S_x *and* S_y*—Category 2 states*

We now consider the mean value for spin components for the Category 2 states. For example, in the case of the spin component S_x,

$$\begin{aligned}\langle S_x\rangle &= \sum_\lambda P(\lambda|c)\langle S_x(\lambda)\rangle \\ &= \frac{1}{2}\sum_\lambda[\langle x_A(\lambda)\rangle\langle x_B(\lambda)\rangle_Q + \langle p_A(\lambda)\rangle\langle p_B(\lambda)\rangle_Q]P(\lambda|c)\end{aligned} \qquad (48)$$

using (44) and (12). This expression involves the hidden variable mean values for the (classical) observables x_A and p_A of subsystem A and the LHS mean values for the quantum quadrature operators $\widehat{x}_B$ and $\widehat{p}_B$. The latter must also correspond to quantum mean values, for a physically realizable quantum state for subsystem B. Thus $\langle x_B(\lambda)\rangle_Q = \mathrm{Tr}(\widehat{x}_B\widehat{\rho}^B(\lambda))$ and $\langle p_B(\lambda)\rangle_Q = \mathrm{Tr}(\widehat{p}_B\widehat{\rho}^B(\lambda))$. Since subsystem B is to be treated quantum mechanically then the density operator $\widehat{\rho}^B(\lambda)$ would be required to both satisfy the *symmetrization principle* and be *local particle number SSR* compliant. Hence there is a constraint based on the LHS $\widehat{\rho}^B(\lambda)$ being a *possible state* for subsystem B that requires the state to be local particle number SSR compliant.

In this case then since both $\widehat{x}_B$ and $\widehat{p}_B$ are just linear combinations of $\widehat{b}$ and $\widehat{b}^\dagger$ we have

$$\langle x_B(\lambda)\rangle_Q = \mathrm{Tr}\frac{1}{\sqrt{2}}(\widehat{b}+\widehat{b}^\dagger)\widehat{\rho}^B(\lambda) = 0,$$

$$\langle p_B(\lambda)\rangle_Q = \mathrm{Tr}\frac{1}{\sqrt{2}i}(\widehat{b}-\widehat{b}^\dagger)\widehat{\rho}^B(\lambda) = 0, \tag{49}$$

$$\langle S_x(\lambda)\rangle = 0, \quad \langle S_y(\lambda)\rangle = 0, \tag{50}$$

and thus for Category 2 states

$$\langle S_x\rangle = 0, \quad \langle S_y\rangle = 0. \tag{51}$$

We do not need to know the outcome for $\langle x_A(\lambda)\rangle$ or $\langle p_A(\lambda)\rangle$.

So that if LHVT is to give the same prediction as quantum theory then on reverting to quantum operators and using (18) we have for Category 2 states

$$\langle\widehat{S}_x\rangle = 0 \quad\text{and}\quad \langle\widehat{S}_y\rangle = 0. \tag{52}$$

These two results are the same as for a quantum separable (Category 1) state.

2. Bloch vector test

From (52) for Category 2 (or Category 1) states we immediately see that if

$$\langle\widehat{S}_x\rangle \neq 0 \quad\text{or}\quad \langle\widehat{S}_y\rangle \neq 0, \tag{53}$$

then the quantum state cannot be in Category 2 (or Category 1). The *Bloch vector test* $\langle\widehat{S}_x\rangle \neq 0$ or $\langle\widehat{S}_y\rangle \neq 0$ now also shows that the state is *EPR steered* as well as just being entangled.

Experiments in two-mode BEC by [17,20] have found nonzero behavior for $\langle\widehat{S}_x\rangle$, $\langle\widehat{S}_y\rangle$. These experiments therefore demonstrate EPR steering, though only entanglement was claimed to have been shown [17]. The application of the Bloch vector test for EPR steering to the experiment in Ref. [20] is discussed more fully elsewhere [28].

E. Spin squeezing tests for EPR steering

1. Mean values of spin component $\mathbf{S}_z$ and number N—Category 2 states

For the other spin component S_z we find using (45) that for the Category 2 states

$$\langle S_z\rangle = \tfrac{1}{2}\langle 1_A\otimes N_B\rangle - \tfrac{1}{2}\langle N_A\otimes 1_B\rangle. \tag{54}$$

As in the quantum separable state case $\langle S_z\rangle$ is not necessarily zero.

2. Variances of spin components $\mathbf{S}_x$ and $\mathbf{S}_y$—Category 2 states

As $\langle S_x(\lambda)\rangle = \langle S_y(\lambda)\rangle = 0$ from (50) we see that $\langle\Delta S_x^2(\lambda)\rangle = \langle S_x^2(\lambda)\rangle$ and $\langle\Delta S_y^2(\lambda)\rangle = \langle S_y^2(\lambda)\rangle$. Using (18), the LHVT expression for S_x^2 obtained from the classical form of (38) and after applying the inequality (C27) we then have the following inequalities for Category 2 states:

$$\langle\Delta S_x^2\rangle \geqslant \sum_\lambda P(\lambda|c)\left\{\frac{1}{4}[\langle x_A^2(\lambda)\rangle\langle x_B^2(\lambda)\rangle_Q + \langle p_A^2(\lambda)\rangle\langle p_B^2(\lambda)\rangle_Q] + \frac{1}{2}[\langle U_A(\lambda)\rangle\langle U_B(\lambda)\rangle_Q - \langle V_A(\lambda)\rangle\langle V_B(\lambda)\rangle_Q]\right\},$$

$$\langle\Delta S_y^2\rangle \geqslant \sum_\lambda P(\lambda|c)\left\{\frac{1}{4}[\langle p_A^2(\lambda)\rangle\langle x_B^2(\lambda)\rangle_Q + \langle x_A^2(\lambda)\rangle\langle p_B^2(\lambda)\rangle_Q] - \frac{1}{2}[\langle U_A(\lambda)\rangle\langle U_B(\lambda)\rangle_Q + \langle V_A(\lambda)\rangle\langle V_B(\lambda)\rangle_Q]\right\}. \tag{55}$$

3. Evaluation of expressions needed—Category 2 states

To consider spin squeezing, spin variance and correlation tests for EPR steering based on the Category 2 states we will need to consider the following additional quantum theory based expressions: $\langle x_B^2(\lambda)\rangle_Q$, $\langle p_B^2(\lambda)\rangle_Q$, $\langle V_B(\lambda)\rangle_Q$, $\langle U_B(\lambda)\rangle_Q$, and the following nonquantum expressions: $\langle x_A^2(\lambda)\rangle$, $\langle p_A^2(\lambda)\rangle$, $\langle V_A(\lambda)\rangle$.

Starting with the *quantum* theory expressions (33) we find that

$$\langle x_B^2(\lambda)\rangle_Q = \mathrm{Tr}((\widehat{b}^\dagger\widehat{b})\widehat{\rho}^B(\lambda)) + \tfrac{1}{2}, \\ = \langle N_B(\lambda)\rangle_Q + \tfrac{1}{2}, \tag{56}$$

$$\langle p_B^2(\lambda)\rangle_Q = \langle N_B(\lambda)\rangle_Q + \tfrac{1}{2}, \tag{57}$$

where the commutation rules have been used and the SSR constraints eliminate the $\mathrm{Tr}((\widehat{b})^2\widehat{\rho}^B(\lambda))$ and $\mathrm{Tr}((\widehat{b}^\dagger)^2\widehat{\rho}^B(\lambda))$ terms. Note that $\langle N_B(\lambda)\rangle_Q \geqslant 0$.

Then using (40) we find that

$$\langle U_B(\lambda)\rangle_Q = \frac{1}{2i}\mathrm{Tr}((\widehat{b})^2 - (\widehat{b}^\dagger)^2)\widehat{\rho}^B(\lambda) = 0, \tag{58}$$

again due to the SSR constraints on the hidden state $\widehat{\rho}^B(\lambda)$.

Also, using (35)

$$\langle V_B(\lambda)\rangle_Q = \tfrac{1}{2}\mathrm{Tr}_B(\widehat{1}_B\widehat{\rho}^B(\lambda)) = \tfrac{1}{2} \tag{59}$$

since the trace of a density operator is unity. Using (56), (57), and (59) we confirm the result that $\langle N_B(\lambda)\rangle_Q = \frac{1}{2}\langle x_B^2(\lambda)\rangle_Q + \frac{1}{2}\langle p_B^2(\lambda)\rangle_Q - \langle V_B(\lambda)\rangle_Q$ consistent with (45). Result (59) also follows directly from (47).

For the *local hidden variable theory* expressions involving subsystem A we have using (45)

$$\langle x_A^2(\lambda)\rangle + \langle p_A^2(\lambda)\rangle = 2\langle N_A(\lambda)\rangle + 2\langle V_A(\lambda)\rangle. \tag{60}$$

Note the analogous result for subsystem B.

Using the results (35) and (55)–(60) we now have for Category 2 states

$$\begin{aligned}\langle\Delta S_x^2\rangle &\geqslant \sum_\lambda P(\lambda|c)\left\{\frac{1}{2}\left[\langle N_B(\lambda)\rangle_Q + \frac{1}{2}\right](\langle N_A(\lambda)\rangle + \langle V_A(\lambda)\rangle) - \frac{1}{4}\langle V_A(\lambda)\rangle\right\},\\ &\geqslant \frac{1}{2}\langle N_A\otimes N_B\rangle + \frac{1}{2}\langle V_A\otimes N_B\rangle + \frac{1}{4}\langle N_A\otimes 1_B\rangle + \frac{1}{4}\langle V_A\otimes 1_B\rangle - \frac{1}{4}\langle V_A\otimes 1_B\rangle,\\ &\geqslant \frac{1}{2}\langle\widehat{N}_A\otimes\widehat{N}_B\rangle + \frac{1}{2}\langle\widehat{V}_A\otimes\widehat{N}_B\rangle + \frac{1}{4}\langle\widehat{N}_A\otimes\widehat{1}_B\rangle,\\ &\geqslant \frac{1}{2}\langle\widehat{N}_A\otimes\widehat{N}_B\rangle + \frac{1}{4}\langle\widehat{1}_A\otimes\widehat{N}_B\rangle + \frac{1}{4}\langle\widehat{N}_A\otimes\widehat{1}_B\rangle,\end{aligned}$$

$$\langle\Delta S_y^2\rangle \geqslant \frac{1}{2}\langle\widehat{N}_A\otimes\widehat{N}_B\rangle + \frac{1}{4}\langle\widehat{1}_A\otimes\widehat{N}_B\rangle + \frac{1}{4}\langle\widehat{N}_A\otimes\widehat{1}_B\rangle. \tag{61}$$

Note that moving from line one to line two only involves LHVT expressions, whereas moving from line two to line three involves replacing the LHVT overall mean values by the equivalent quantum expressions, and in the next line the quantum operator $\widehat{V}_A$ is replaced by $\widehat{1}_A/2$. These inequalities are the same as those for Category 1 states (see Ref. [3]). Note that the SSRs for the LHS have been used in deriving these last results. Also from (54),

$$\tfrac{1}{2}|\langle S_z\rangle| \leqslant \tfrac{1}{4}\langle 1_A \otimes N_B\rangle + \tfrac{1}{4}\langle N_A \otimes 1_B\rangle. \tag{62}$$

The last line follows from the LHVT expression $\langle 1_A \otimes N_B\rangle$ giving the mean number of bosons in mode B and for this to be the same as the quantum theory expression $\langle \widehat{1}_A \otimes \widehat{N}_B\rangle$. As the eigenvalues of the number operator $\widehat{N}_B = \widehat{b}^\dagger \widehat{b}$ are never negative $\langle \widehat{1}_A \otimes \widehat{N}_B\rangle$ and hence $\langle 1_A \otimes N_B\rangle$ is never negative, so $|\langle 1_A \otimes N_B\rangle| = \langle 1_A \otimes N_B\rangle$. Similarly, $\langle N_A \otimes 1_B\rangle$ is never negative. This result is the same as that for Category 1 states (see Ref. [3]).

Combining (61) and (62) we find using LHVT that for Category 2 states

$$\begin{aligned} \langle \Delta S_x^2\rangle - \tfrac{1}{2}|\langle S_z\rangle| &\geqslant \tfrac{1}{2}\langle N_A \otimes N_B\rangle, \\ \langle \Delta S_y^2\rangle - \tfrac{1}{2}|\langle S_z\rangle| &\geqslant \tfrac{1}{2}\langle N_A \otimes N_B\rangle, \end{aligned} \tag{63}$$

so as the LHVT is required to predict the same results as for quantum theory we have for Category 2 states

$$\langle \Delta \widehat{S}_x^2\rangle - \tfrac{1}{2}|\langle \widehat{S}_z\rangle| \geqslant \tfrac{1}{2}\langle \widehat{N}_A \otimes \widehat{N}_B\rangle \geqslant 0, \tag{64}$$

$$\langle \Delta \widehat{S}_y^2\rangle - \tfrac{1}{2}|\langle \widehat{S}_z\rangle| \geqslant \tfrac{1}{2}\langle \widehat{N}_A \otimes \widehat{N}_B\rangle \geqslant 0. \tag{65}$$

The expression $\frac{1}{2}\langle \widehat{N}_A \otimes \widehat{N}_B\rangle$ is never negative because the eigenvalues of $\widehat{N}_A$ and $\widehat{N}_B$ are never negative.

4. Spin squeezing tests

From Eq. (52) we immediately see that for a quantum state where the observable $\widehat{S}_z$ is squeezed with respect to $\widehat{S}_x$ or with respect to $\widehat{S}_y$, then it cannot be a Category 2 state, because spin squeezing in $\widehat{S}_z$ requires $\langle \Delta \widehat{S}_z^2\rangle$ to be less than either $|\langle \widehat{S}_x\rangle|/2$ or $|\langle \widehat{S}_y\rangle|/2$ and this is impossible for both Category 1 (see Ref. [3]) and Category 2 states—where $\langle \widehat{S}_x\rangle = \langle \widehat{S}_y\rangle = 0$. This condition also rules out $\widehat{S}_x$ or $\widehat{S}_y$ being squeezed with respect to $\widehat{S}_z$, or $\widehat{S}_z$ being squeezed with respect to $\widehat{S}_x$ or $\widehat{S}_y$. In Ref. [3] it was shown that spin squeezing involving $\widehat{S}_z$ provided a test for entanglement. Here we see that spin squeezing involving the observable $\widehat{S}_z$ shows the state is *EPR steered* as well as merely being *entangled*.

From Eqs. (64) and (65) we see that for Category 2 states $(\langle \Delta \widehat{S}_x^2\rangle - \frac{1}{2}|\langle \widehat{S}_z\rangle|) \geqslant 0$ and $(\langle \Delta \widehat{S}_y^2\rangle - \frac{1}{2}|\langle \widehat{S}_z\rangle|) \geqslant 0$. Hence we find that for Category 2 states there is no spin squeezing in $\widehat{S}_x$ compared to $\widehat{S}_y$ (or vice versa). For Category 1 states we also find that $(\langle \Delta \widehat{S}_x^2\rangle - \frac{1}{2}|\langle \widehat{S}_z\rangle|) \geqslant \frac{1}{2}\langle \widehat{N}_A \otimes \widehat{N}_B\rangle \geqslant 0$ and $(\langle \Delta \widehat{S}_y^2\rangle - \frac{1}{2}|\langle \widehat{S}_z\rangle|) \geqslant \frac{1}{2}\langle \widehat{N}_A \otimes \widehat{N}_B\rangle \geqslant 0$ (see Eq. (31) in Ref. [3]). Hence spin squeezing in $\widehat{S}_x$ versus $\widehat{S}_y$ (or vice versa) is a test for entanglement, so the state is *not* in Category 1. Thus spin squeezing in $\widehat{S}_x$ versus $\widehat{S}_y$ (or vice versa) is therefore also a test for *EPR steering*.

Overall then we now see that *spin squeezing* in *any* spin component $\widehat{S}_\alpha$ with respect to another component $\widehat{S}_\beta$,

$$\langle \Delta \widehat{S}_\alpha^2\rangle < \tfrac{1}{2}|\langle \widehat{S}_\gamma\rangle| \quad \text{and} \quad \langle \Delta \widehat{S}_\beta^2\rangle > \tfrac{1}{2}|\langle \widehat{S}_\gamma\rangle|, \tag{66}$$

(where α, β, γ are x, y, z in cyclic order) is a sufficiency test for *EPR steering*. Hence *spin squeezing* in *any* spin component $\widehat{S}_\alpha$ with respect to another component $\widehat{S}_\beta$ shows that the state is *EPR steered* as well as just being entangled.

Experiments in two-mode BEC by [17–19] have found spin squeezing in S_z. These experiments therefore demonstrate EPR steering, though only entanglement was claimed to have been shown in Refs. [17,18].

F. Planar spin variance tests for EPR steering

1. Mean values of total boson number N—Category 2 states

For the number observable N we have from (45)

$$\langle N\rangle = \langle 1_A \otimes N_B\rangle + \langle N_A \otimes 1_B\rangle. \tag{67}$$

This result is the same as that for Category 1 states (see Ref. [3]).

2. Hillery-Zubairy planar spin variance test

The Hillery-Zubairy spin variance test [14] for quantum entanglement is $\langle \Delta \widehat{S}_x^2\rangle + \langle \Delta \widehat{S}_y^2\rangle - \frac{1}{2}\langle \widehat{N}\rangle < 0$. We now consider the quantity $\langle \Delta S_x^2\rangle + \langle \Delta S_y^2\rangle - \frac{1}{2}\langle N\rangle$ for Category 2 states using the results based on LHVT in Eqs. (61) and (67). We find that

$$\langle \Delta S_x^2\rangle + \langle \Delta S_y^2\rangle - \tfrac{1}{2}\langle N\rangle \geqslant \langle N_A \otimes N_B\rangle \geqslant 0. \tag{68}$$

Thus if LHVT is to predict the same result as quantum theory it follows that for Category 2 states that

$$\langle \Delta \widehat{S}_x^2\rangle + \langle \Delta \widehat{S}_y^2\rangle - \tfrac{1}{2}\langle \widehat{N}\rangle \geqslant 0. \tag{69}$$

This result also applies for Category 1 states [see Eqs. (82) and (83) in Ref. [3] for details, or directly from Eq. (I15)].

Hence we can say that if

$$\langle \Delta \widehat{S}_x^2\rangle + \langle \Delta \widehat{S}_y^2\rangle - \tfrac{1}{2}\langle \widehat{N}\rangle < 0, \tag{70}$$

then the state is not in Category 2. It also shows that it is not in Category 1 (separable states), this being the Hillery-Zubairy planar spin variance test [14] for entanglement. This condition can also be written as

$$E_{\mathrm{HZ}} = \frac{\langle \Delta \widehat{S}_x^2\rangle + \langle \Delta \widehat{S}_y^2\rangle}{\frac{1}{2}\langle \widehat{N}\rangle} < 1, \tag{71}$$

which is the form given in Ref. [15].

Hence the *Hillary-Zubairy planar spin variance* inequality is a sufficiency test for *EPR steering* as well as demonstrating entanglement.

3. Generalized Hillery-Zubairy planar spin variance test

The results (61), (67), and (54) show that for Category 2 states where the LHS occurs in subsystem B:

$$\begin{aligned} &\langle \Delta S_x^2\rangle + \langle \Delta S_y^2\rangle - \tfrac{1}{4}\langle N\rangle + \tfrac{1}{2}\langle S_z\rangle \\ &\quad\geqslant \langle N_A \otimes N_B\rangle + \tfrac{1}{2}\langle 1_A \otimes N_B\rangle, \\ &\quad\geqslant 0. \end{aligned} \tag{72}$$

The details are set out in Appendix G.

This provides a generalization of the *Hillery-Zubairy planar spin variance* test [14] for *EPR steering*. In the case we see that if

$$\langle\Delta\widehat{S}_x^2\rangle+\langle\Delta\widehat{S}_y^2\rangle-\tfrac{1}{4}\langle\widehat{N}\rangle+\tfrac{1}{2}\langle\widehat{S}_z\rangle<0, \tag{73}$$

then the state is not in Category 2. If subsystem A involves the LHS, then $+\frac{1}{2}\langle\widehat{S}_z\rangle$ is replaced by $-\frac{1}{2}\langle\widehat{S}_z\rangle$. Since $+\frac{1}{2}\langle\widehat{N}\rangle\geqslant\langle\widehat{S}_z\rangle\geqslant-\frac{1}{2}\langle\widehat{N}\rangle$ then $\frac{1}{2}\langle\widehat{N}\rangle\geqslant\frac{1}{4}\langle\widehat{N}\rangle+\frac{1}{2}\langle\widehat{S}_z\rangle\geqslant 0$, so as $\langle\Delta\widehat{S}_x^2\rangle+\langle\Delta\widehat{S}_y^2\rangle-\frac{1}{4}\langle\widehat{N}\rangle+\frac{1}{2}\langle\widehat{S}_z\rangle=\langle\Delta\widehat{S}_x^2\rangle+\langle\Delta\widehat{S}_y^2\rangle-\frac{1}{2}\langle\widehat{N}\rangle+(\frac{1}{4}\langle\widehat{N}\rangle+\frac{1}{2}\langle\widehat{S}_z\rangle)$ and we have just shown that $(\frac{1}{4}\langle\widehat{N}\rangle+\frac{1}{2}\langle\widehat{S}_z\rangle)$ is never negative, then if (73) is satisfied, then the Hillary-Zubairy planar spin variance test in (70) must also apply, showing (see Ref. [3] for details) that the state cannot be in Category 1. The latter test is of course itself sufficient to demonstrate EPR steering. Since $0\leqslant\frac{1}{4}\langle\widehat{N}\rangle-\frac{1}{2}\langle\widehat{S}_z\rangle\leqslant\frac{1}{2}\langle\widehat{N}\rangle$ it is of course harder to find states where $\langle\Delta\widehat{S}_x^2\rangle+\langle\Delta\widehat{S}_y^2\rangle<\frac{1}{4}\langle\widehat{N}\rangle-\frac{1}{2}\langle\widehat{S}_z\rangle$ to show EPR steering than merely being less than $\frac{1}{2}\langle\widehat{N}\rangle$, as would also show EPR steering. The generalized Hillery-Zubairy planar spin variance test (73) for EPR steering is a more difficult test to satisfy than the Hillery-Zubairy test. In the generalized form (73) the EPR steering test now allows for *asymmetry* $(\langle\widehat{S}_z\rangle\neq 0)$.

The generalized Hillery-Zubairy EPR steering test in (73) can also be written as

$$E_{\mathrm{GHZ}}=\frac{\langle\Delta\widehat{S}_x^2\rangle+\langle\Delta\widehat{S}_y^2\rangle}{\frac{1}{2}\langle\widehat{N}\rangle}<\frac{\langle\widehat{N}_A\rangle}{\langle\widehat{N}\rangle} \tag{74}$$

after substituting $\langle\widehat{N}\rangle=\langle\widehat{N}_A\rangle+\langle\widehat{N}_B\rangle$ and $\langle\widehat{S}_z\rangle=(\langle\widehat{N}_B\rangle-\langle\widehat{N}_A\rangle)/2$, which is consistent with the result $E_{\mathrm{HZ}}<1/2$ previously obtained by He *et al.* in Ref. [15] for $\langle\widehat{S}_z\rangle=0$. This form of the test also shows that the EPR steering test in (71) is satisfied, since the right side is always less than unity because $\langle\widehat{N}_A\rangle\leqslant\langle\widehat{N}\rangle$. Note that for EPR steering to apply, it is not *necessary* that (74) applies, since (71) is *sufficient* to demonstrate EPR steering. Combining both tests we see that if *either* ($E_{\mathrm{HZ}}<1$ *and* $E_{\mathrm{GHZ}}<\langle\widehat{N}_A\rangle/\langle\widehat{N}\rangle$) *or* ($E_{\mathrm{HZ}}<1$) then the state cannot be either Category 1 or Category 2, and hence is EPR steerable.

The tests in (73) and (74) also follow from the strong correlation condition obtained by Cavalcanti *et al.* [16]—set out here as Eq. (I11) (see Appendices I and J). The derivation of the test (73) in terms of spin operators starting from the strong correlation condition (I11) is set out in Appendix I 3. The test given in (74) was first stated in Ref. [29], again starting from the strong correlation condition in Ref. [16], and then expressing the latter inequality in terms of spin operators—as derived here in Appendix I 3.

These two planar spin variance test are involved in discussing the so-called *depth of EPR steering* in two-mode BECs [29], which specifies the number of particles involved in the component of the density operator which is responsible for EPR steering effects.

G. Two-mode quadrature squeezing test for EPR steering

1. Mean values for two-mode quadratures $X_\theta(\pm)$ and $P_\theta(\pm)$—Category 2 states

We now consider the mean value for two-mode quadrature observables for the Category 2 states. For example, in the case of the quadratures $X_\theta(\pm)$,

$$\begin{aligned}\langle X_\theta(\pm)\rangle=&\frac{1}{\sqrt{2}}\sum_\lambda P(\lambda|c)[\langle x_A(\lambda)\rangle\cos\theta+\langle p_A(\lambda)\rangle\sin\theta\\&\pm\langle x_B(\lambda)\rangle_Q\cos\theta\pm\langle p_B(\lambda)\rangle_Q\sin\theta],\end{aligned} \tag{75}$$

using Eq. (46). A similar result is found for $P_\theta(\pm)$. We then use the previous results (49) for subsystem B to find

$$\begin{aligned}\langle X_\theta(\pm)\rangle&=\frac{1}{\sqrt{2}}\sum_\lambda P(\lambda|c)[\langle x_A(\lambda)\rangle\cos\theta+\langle p_A(\lambda)\rangle\sin\theta],\\\langle P_\theta(\pm)\rangle&=\frac{1}{\sqrt{2}}\sum_\lambda P(\lambda|c)[-\langle x_A(\lambda)\rangle\sin\theta+\langle p_A(\lambda)\rangle\cos\theta].\end{aligned} \tag{76}$$

2. Variances for two-mode quadratures—Category 2 states

Using (18) and the LHVT expression for $X_\theta(\pm)^2$ obtained from the equivalent of Eq. (43) for classical observables we have for Category 2 states,

$$\begin{aligned}\langle X_\theta(\pm)^2\rangle=&\frac{1}{2}\sum_\lambda P(\lambda|c)\big[\langle x_A^2(\lambda)\rangle\cos^2\theta+\langle U_A(\lambda)\rangle 2\sin\theta\cos\theta\\&+\langle p_A^2(\lambda)\rangle\sin^2\theta\big]+\frac{1}{2}\sum_\lambda P(\lambda|c)\left[\langle N_B(\lambda)\rangle_Q+\frac{1}{2}\right],\end{aligned} \tag{77}$$

where we have used the previous results (49) and (58) for subsystem B to eliminate terms involving $\langle x_B(\lambda)\rangle_Q$, $\langle p_B(\lambda)\rangle_Q$ and $\langle U_B(\lambda)\rangle_Q$ and the results (56) and (57) for $\langle x_B^2(\lambda)\rangle_Q$ and $\langle p_B^2(\lambda)\rangle_Q$ to simplify the last term.

We next use the LHVT–quantum theory equivalences (19) to replace (76) and (77) by their quantum forms. Quantum forms for the variances are then obtained. Finally we use the result from Sec. II A the *reduced density operator* for subsystem A satisfies the local particle number SSR to obtain expressions for $\langle x_A\rangle$, $\langle p_A\rangle$, $\langle x_A^2\rangle$, $\langle p_A^2\rangle$, and $\langle U_A\rangle$ to give the following results for the variances $\langle\Delta X_\theta(\pm)^2\rangle$ and $\langle\Delta P_\theta(\pm)^2\rangle$ for Category 2 states [see Eq. (H7)]:

$$\begin{aligned}\langle\Delta X_\theta(\pm)^2\rangle&=\tfrac{1}{2}\langle N\rangle+\tfrac{1}{2}\geqslant\tfrac{1}{2},\\\langle\Delta P_\theta(\pm)^2\rangle&=\tfrac{1}{2}\langle N\rangle+\tfrac{1}{2}\geqslant\tfrac{1}{2}.\end{aligned} \tag{78}$$

Details are given in Appendix H. The same results apply for Category 1 (separable) states (see Appendix L in Ref. [3]).

3. Two-mode quadrature squeezing test

We have shown for Category 2 states (see Eq. (78)) that $\langle\Delta X_\theta(\pm)^2\rangle=\langle\Delta P_\theta(\pm)^2\rangle=\frac{1}{2}\langle N\rangle+\frac{1}{2}$, and the right side is never less than one half. The same result applied for Category 1 states. Hence it follows that if

$$\langle\Delta\widehat{X}_\theta(\pm)^2\rangle<\tfrac{1}{2}\quad\text{or}\quad\langle\Delta\widehat{P}_\theta(\pm)^2\rangle<\tfrac{1}{2}, \tag{79}$$

which is the condition for *squeezing* in either of the *two-mode quadrature* observables $X_\theta(\pm)$ or $P_\theta(\pm)$, then the state is not in Category 1 or 2. Due to the Heisenberg uncertainty principle $\langle\Delta\widehat{X}_\theta(\pm)^2\rangle\langle\Delta\widehat{P}_\theta(\pm)^2\rangle \geqslant 1/4$ only one of the pair of quadrature operators is squeezed. Thus *two-mode quadrature squeezing* as in (79) provides a sufficiency test for *EPR steering*.

Experiments in two-mode BEC in Refs. [21,22] have found two-mode quadrature squeezing in S_z. These experiments therefore demonstrate EPR steering, which was identified in these papers.

H. Two-mode binomial state

The two-mode *binomial* state given by

$$|\Phi\rangle = \frac{[(\widehat{a}^\dagger + \widehat{b}^\dagger)/\sqrt{2}]^N}{\sqrt{N!}}|0\rangle \tag{80}$$

provides for a simple illustration of some of the EPR steering tests. Results for mean values and variances of the spin operators $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ and number operators $\widehat{N}_A$, $\widehat{N}_B$, $\widehat{N}$ are as follows:

$$\begin{aligned}
&\langle\widehat{N}\rangle = N, && \langle\widehat{N}_A\rangle = \frac{N}{2}, && \langle\widehat{N}_B\rangle = \frac{N}{2},\\
&\langle\widehat{S}_x\rangle = \frac{N}{2}, && \langle\widehat{S}_y\rangle = 0, && \langle\widehat{S}_z\rangle = 0,\\
&\langle\Delta\widehat{S}_x^2\rangle = 0, && \langle\Delta\widehat{S}_y^2\rangle = \frac{N}{4}, && \langle\Delta\widehat{S}_z^2\rangle = \frac{N}{4},
\end{aligned} \tag{81}$$

(see Ref. [3] for details). From these results we see that

$$\begin{aligned}
\langle\widehat{S}_x\rangle &\neq 0,\\
\langle\Delta\widehat{S}_y^2\rangle - \frac{1}{2}|\langle\widehat{S}_x\rangle| &= 0,\\
\langle\Delta\widehat{S}_z^2\rangle - \frac{1}{2}|\langle\widehat{S}_x\rangle| &= 0,\\
E_{\mathrm{HZ}} &= \frac{1}{2} < 1,\\
E_{\mathrm{GHZ}} &= \frac{1}{2} = \frac{\langle\widehat{N}_A\rangle}{\langle\widehat{N}\rangle}.
\end{aligned} \tag{82}$$

Hence the Bloch vector test and the Hillery-Zubairy planar spin variance test both predict EPR steering, though neither the spin squeezing test or the generalized Hillery-Zubairy planar spin variance test does this. Nevertheless, EPR steering does occur for this state, since we only require one of the tests to be positive. That the state is steerable in the EPR sense may be seen if the measurables for the two modes are the number operators $\widehat{N}_A$, $\widehat{N}_B$. The measurement of $\widehat{N}_A$ leading to the outcome n_A changes the quantum state to be the number state $(\widehat{a}^\dagger)^{n_A}(\widehat{b}^\dagger)^{N-n_A}|0\rangle/(\sqrt{n_A!}\sqrt{(N-n_A)!})$, so that measurement of $\widehat{N}_B$ must lead to the outcome $N-n_A$ in accordance with EPR steering.

V. SUMMARY AND CONCLUSION

Tests for EPR steering (EPR entanglement) based on violation of the LHS model have been examined for two-mode systems of identical massive bosons, such as occur in BECs. Such tests were obtained based on whether the Bloch vector is in the xy plane (Bloch vector test) and on whether there is spin squeezing in any of the spin components S_x, S_y, or S_z (spin squeezing test). Experiments that have been carried out on two-mode BEC [17–22] have demonstrated EPR steering in such two-mode systems. The Hillery planar spin variance test based on the sum of variances in S_x and S_y also demonstrates EPR steering. In addition, two-mode quadrature squeezing also provides a test for EPR steering. A generalized Hillery-Zubairy planar spin variance test for EPR steering was found, involving the sum of variances in S_x and S_y, but now containing a different multiple of the mean value for N along with a term involving the mean value for S_z. This allows for asymmetry and is a stronger version of the Hillery planar spin variance test. Correlation tests based on the mean value of $\langle a^\dagger b\rangle$ have also been obtained by others [16], and these are equivalent to some of the tests based on the spin operators. No EPR steering test based on the difference between the variances of the number difference and number sum was found. We note that some of the tests (Bloch vector, spin squeezing, two-mode quadrature squeezing) were based on applying the SSRs for the total particle number as well as that for the local particle number for the subsystem LHS. However, since the stronger correlation inequalities from which they can also be derived do not depend on the SSR (see Appendix I 2) the Hillery-Zubairy planar spin variance test and its generalization involving the mean value for S_z do not depend on these rules.

The treatment involved considering two possible classification schemes for the quantum states of bipartite composite systems. In the first (quantum theory classification scheme) the states are classified as being either quantum separable or quantum entangled. In the second (local hidden variable theory classification scheme) the states are initially classified as being Bell local or Bell nonlocal. The Bell nonlocal states are quantum entangled and EPR steerable—these are listed as Category 4 states. However, the Bell local states can be divided up into three categories depending on whether both, one or neither of the subsystem single measurement probability is given by a quantum theory expression involving a subsystem density operator. The Category 1 states (both) are the same as the quantum separable states and are nonentangled, LHSs, and nonsteered. The Category 2 states (one) are quantum entangled LHSs and are nonsteerable. The Category 3 (neither) states are quantum entangled and EPR steerable. A detailed study of how observables are treated in terms of quantum theory and local hidden variable theories was also carried out, including how the two approaches are related and how to replace quantum operators for observables with classical entities. For systems involving identical bosons the mode annihilation, creation operators are replaced by quadrature amplitudes. Certain auxiliary observables also needed to be introduced.

In a later paper we will consider tests for *Bell nonlocality* that can be applied when the measurable quantities for the two subsystems have a range of outcomes other than the more limited +1, −1 outcomes considered by Clauser *et al.* [27].

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APPENDIX A: REVIEW OF HIDDEN VARIABLE THEORY AND QUANTUM STATES

1. Origin of hidden variable theory

Local hidden variable theory has its origins in papers by Einstein, Schrödinger, Bell, and Werner ([4,5,10–12]). Einstein suggested that quantum theory, though correctly predicting the probabilities for measurement outcomes was nevertheless an *incomplete* theory—in that the probabilistic measurement outcomes predicted in quantum theory could just be the *statistical* outcome of an underlying *deterministic* theory, where the possible measured outcomes for all observables *always* have specific values irrespective of whether an *actual* measurement has taken place. Hence possible outcomes for observable quantities (such as position and momentum) could always be regarded as elements of *reality* independent of measurement The EPR paradox is based on this assumption and involved an entangled state for two *well-separated* and no longer interacting distinguishable particles, which had well-defined values for the position *difference* and the momentum *sum*. Because of these correlations, the choice of measuring the position (or the momentum) for the first particle would instantly determine the outcome for the position (or the momentum) of the second particle—a feature we now refer to as *steering*—but which Einstein called "spooky action at a distance" because it conflicted with causality (since no signal would have had time to travel between the two particles). The paradox is that by measuring (for example) the position for the first particle, we then know the position for the second particle *without* doing a measurement, so by then measuring the momentum for the second particle a joint precise measurement of *both* the position and momentum for the second particle would have occurred—which evidently conflicts with the Heisenberg uncertainty principle. Bohm [30] described a similar paradox to EPR, but now involving a system consisting of two spin $1/2$ particles in a singlet state, and where the observables were spin components with *quantized* measured outcomes rather than the *continuous* outcomes that applied to EPR. The Schrödinger cat paradox [11] is another example, but now involving a *macroscopic* subsystem (the cat) in an entangled state with a *microscopic* subsystem (the two-state radioactive atom). From the Einstein concept of reality, the cat must be *either* alive *or* dead even *before* the box is opened to see what is the case. However, from the Copenhagen interpretation of quantum theory (see Ref. [31] for a discussion), the values for observables do not have a presence in reality *until* measurement takes place. Hence from the Copenhagen viewpoint the cat is *neither* dead *nor* alive *until* the box is opened. Similarly, in the EPR experiment the second particle does not have a position (or momentum) until the observable is measured. Reality thus *emerges* as the result of measurement. Thus from the Copenhagen perspective of what constitutes reality, there are no paradoxes in either the EPR or Schrödinger cat scenarios.

Einstein believed that an underlying *realist* theory could be found, based on what are now referred to as *hidden variables*—which would specify the real or underlying state of the system. Thus, quantum theory is not wrong, it is merely *incomplete*. However, it was not until 1965 before a quantitative general form for *local hidden variable theory* was proposed by Bell [4]. This was relevant for the EPR paradox and could be tested in experiments. In its simplest form, the key idea is that hidden variables are specified probabilistically when the state for the composite system is prepared, and these would determine the *actual* values for *all* the subsystem observables even after the subsystems have separated—and even if the observables were *incompatible* with *simultaneous* precise measurements according to quantum theory (such as two different spin components). In the EPR experiment the hidden variables would specify *both* the position and momentum for each distinguishable particle. More elaborate versions of local hidden variable theory only require the hidden variables to determine the *probabilities* of measurement outcomes for each of the separate subsystems, with the overall expressions for the joint subsystem measurement outcomes then being obtained in accordance with classical probability theory (see Refs. [2,6,9] and Sec. III for details). Quantum states for composite systems that could be described by local hidden variable theory are referred to as *Bell local*. Quantum states for composite systems that could be described by local hidden variable theory were such that certain inequalities would apply involving the mean values of products for the results of measuring pairs of observables for the two subsystems—the *Bell inequalities* [4,32]. States for which a local hidden variable theory does not apply (and hence do not satisfy Bell inequalities) are the *Bell nonlocal* states. Based on the entangled singlet state of two spin $1/2$ particles Clauser *et al.* [27] proposed an experiment that could demonstrate a violation of a Bell inequality. This showed that local hidden variable theory could not account for an experiment which was explained by quantum theory. Subsequent experimental work violating Bell inequalities confirmed that there are other quantum states for which a local hidden variable theory does *not* apply, and where quantum theory was needed to explain the results (see Brunner *et al.* [33] for a recent review). Numerous loopholes preventing LHVT being ruled out were shown not to apply. However, the existence of *some* quantum states (such as the two qubit singlet *Bell states* [34]) for which the Bell inequalities are *not* obeyed and where the results were confirmed experimentally to agree with quantum theory, is itself sufficient to show that Einstein's hope that an underlying reality represented by a local hidden variable theory could *always* underpin quantum theory *cannot* be realized.

In spite of this, there has been continued interest in determining the circumstances in which the ideas of Einstein, Bell, and others could not be applied—that the predictions of quantum theory are correct, and the experimental results could not be explained by a hidden variable theory. However, experience has shown that finding Bell inequality violations is not easy. Such research is important because it enables the regimes in which quantum theory *must* be applied to be better understood—for example, what states for *macroscopic* systems are Bell nonlocal? And even for states that are Bell local, which of them exhibit the feature of EPR steering? Although not ruling out local hidden variable theory, EPR steering is itself a strange effect in terms of Einstein's viewpoint on reality, so it is of interest to identify circumstances where it occurs. For this research program *bipartite* systems are often studied due to their relative simplicity, and the simplest of these would just involve *two modes*. Since its origins HVT has been focused on the *probabilistic* predictions of quantum theory. However, it should be noted that no *unique* form for HVT has been found that satisfies the constraint of agreeing with *every* feature of quantum theory, even for states and measurement choices where *some* of the predictions agree. As well as being probabilistic, such features include the *quantization* for measured outcomes of certain observables (such as angular momentum components), Heisenberg *uncertainty principle* requirements for the variances of pairs of incompatible observables (such as position and momentum), the presence in quantum theory of observables with *nonclassical* counterparts (such as parity), the existence of a *classical regimes* in quantum theory—as well as general effects such as *quantum interference*. Although it may be possible to find versions of HVT that account for some of these general quantum features, testing whether HVT can account more generally for quantum results is best done via the study of phenomena for which the predictions of HVT and quantum theory are unambiguously different, and cannot be made to agree via minor changes to the details in HVT. It is here that the role of measurements such as *Bell tests* are particularly important, since *Bell inequality violations* rule out *all* versions of at least *local* HVT (though not excluding *nonlocal* forms of HVT where the hidden variables do not determine probabilities for the subsystems separately). As we will see, *spin squeezing* for two-mode systems implies EPR steering, and hence at least ruling out some forms of LHVT—namely, those involving Category 1 and Category 2 LHVT states (see below).

2. Categories of quantum states—Overview

It was recognized [12] that *all* separable states could be described by hidden variable theory (and hence are Bell local) and hence a state had to be *entangled* to be Bell nonlocal. However, Werner [12] showed that *some* entangled states could also be described by hidden variable theory—and hence not violate a Bell inequality. The relationship between the classification of states into separable or entangled on one hand, and a classification into Bell local and Bell nonlocal states on the other hand is therefore not a simple one. This issue will be discussed in detail in Sec. III. In addition to Bell locality or nonlocality, there is the question of which categories of states demonstrate the feature of steering [5,10,11], in which a choice of measurement on one subsystem can be used to instantly affect the outcomes for possible measurements on the other subsystem—even if it they are well separated. For separable states, both subsystem states are specified by quantum density operators which are determined probabilistically in the preparation process. These are examples of the general concept of *local hidden states* (see Refs. [2,6–9])—which are subsystem quantum states whose density operator is specified by hidden variables. Steerability requires the absence of LHSs. The physical reason for this is described in Refs. [6–8], but for completeness this is set out in Appendix F.

In the work by Wiseman *et al.* [6–8] states for bipartite systems defined in terms of local hidden variable theory were first categorized by whether they are Bell local or Bell nonlocal. Within the states that are Bell local a more detailed categorization was made based on a hierarchy of *nondisjoint* subsets—first, by whether they are EPR steerable or not, and then, second, for EPR nonsteerable states by whether they are separable or not. In the present paper we apply the concept of local hidden quantum states (whose density operators are determined from the hidden variables) that were introduced by Wiseman *et al.* to propose a *different* categorization of the Bell local states into three subsets which are *disjoint*. These are related to the hierarchy of nondisjoint subsets introduced by Wiseman *et al.* The disjoint subsets of states are defined by whether two, one or none of the subsystem hidden variable probabilities is *also* obtained from a local hidden quantum state. *Category 1* states involve two hidden states, and this Bell local subset is the same as the separable states. These are nonsteerable. *Category 2* states involve only one hidden state and for this Bell local subset the states are entangled, though nonsteerable. *Category 3* states do not involve any hidden state, and these Bell local states are both entangled and steerable. We will also designate the states that are Bell nonlocal as *Category 4* states, and these states are both entangled and steerable. The categorization of the quantum states both in terms of entanglement versus separability and alternatively Bell locality versus Bell nonlocality is summarized in Fig. 1.

It is of some interest to devise tests for which specific category a quantum state falls into in the context of *bipartite* systems of *identical massive bosons*, such as occur in Bose-Einstein condensates for cold bosonic atomic gases. We treat the simplest situation where each subsystem involves just a *single* mode. For these systems, both the *symmetrization principle* and the *super-selection rule* for particle number must be applied. The focus of this paper is on whether the quantum state is *EPR steerable*—which means showing that it is not a Category 1 or a Category 2 state. In previous work tests have been obtained (see Ref. [3] for details of a range of tests found by various authors) for showing that a state is *entangled*, which therefore rules them out from being in Category 1. Hence we need only to consider tests for showing that the state is also not in Category 2. Based on local hidden variable theory, predictions can be made for Category 2 states involving the mean values and variances for measurement outcomes. For observables associated with the subsystem for which there is a LHS, quantum expressions may be applied.

APPENDIX B: BASIC MEASUREMENT PROBABILITIES FOR BIPARTITE SYSTEMS

This paper deals with measurements on *bipartite* composite quantum systems, where we have two *distinguishable* subsystems A and B which are each associated with measurable physical *observables* Ω_A and Ω_B for which possible *outcomes* are denoted α and β. The composite system exists in various *quantum states*, whose *preparation* is symbolized by c. Quantum theory has the key feature that such measurements the occurrence of particular outcomes are specified by *probabilities* rather than being *deterministic*, and the basic quantity of interest is the *joint probability* $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ for measurement of *any* pair of subsystem *observables* Ω_A and Ω_B to obtain *any* of their possible *outcomes* α and β when the *preparation* process is c. As the subsystems are distinct *simultaneous precise measurement* outcomes apply for the pairs of observables Ω_A and Ω_B in both quantum and hidden variable theory (in the latter case the observables are classical variables and not Hermitian operators). The probability $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ is of course *real* and *positive* and its sum for all outcomes for both Ω_A and Ω_B is equal to unity. The sum of the joint probability over the possible outcomes α for measuring Ω_A *defines* the *single probability* $P(\beta|\Omega_B,c)$ for measuring Ω_B with outcome β, *irrespective* of the outcome for measuring Ω_A. A similar definition applies for the single probability $P(\alpha|\Omega_A,c)$ for measuring Ω_A with outcome α, irrespective of the outcome for measuring Ω_B. Thus,

$$\sum_{\alpha,\beta} P(\alpha,\beta|\Omega_A,\Omega_B,c)=1, \tag{B1}$$

$$P(\beta|\Omega_B,c)=\sum_{\alpha} P(\alpha,\beta|\Omega_A,\Omega_B,c), \tag{B2}$$

$$P(\alpha|\Omega_A,c)=\sum_{\beta} P(\alpha,\beta|\Omega_A,\Omega_B,c). \tag{B3}$$

The single probabilities also satisfy the expected *probability sum rules*

$$\sum_{\beta} P(\beta|\Omega_B,c)=1, \quad \sum_{\alpha} P(\alpha|\Omega_A,c) \quad =1, \tag{B4}$$

which follow from (B1).

From the joint measurement probability $P(\alpha,\beta|\Omega_A,\Omega_B,c)$ and the single measurement probabilities $P(\alpha|\Omega_A,c)$ and $P(\beta|\Omega_B,c)$, we can introduce *conditional probabilities* $P(\beta|\Omega_B||\alpha,\Omega_A,c)$ and $P(\alpha|\Omega_A||\beta,\Omega_B,c)$. Here $P(\beta|\Omega_B||\alpha,\Omega_A,c)$ is the probability that measurement of the observable Ω_B yields the outcome β given that measurement of the observable Ω_A yields the outcome α. This [and the corresponding expression for $P(\alpha|\Omega_A||\beta,\ \Omega_B,c)$] is given by *Bayes' theorem* as

$$\begin{aligned} P(\beta|\Omega_B||\alpha,\Omega_A,c) &= \frac{P(\alpha,\beta|\Omega_A,\Omega_B,c)}{P(\alpha|\Omega_A,c)},\\ P(\alpha|\Omega_A||\beta,\Omega_B,c) &= \frac{P(\alpha,\beta|\Omega_A,\Omega_B,c)}{P(\beta|\Omega_B,c)}. \end{aligned} \tag{B5}$$

All these expressions apply irrespective of whether the joint and single measurement probabilities are obtained from *quantum theory* or *local hidden variable theory* formulas.

APPENDIX C: MEAN VALUES AND VARIANCES—GENERAL FEATURES

1. Mean values and variances—Quantum models

In a fully quantum treatment, any observable represented by a Hermitian operator $\widehat{\Omega}$, whose measured outcomes are its eigenvalues θ, can be written as $\widehat{\Omega}=\sum_\theta \theta\,\widehat{\Pi}_\theta$ in terms of its projectors $\widehat{\Pi}_\theta$ and we can determine the probability $P(\widehat{\Omega},\theta)$ for the outcome θ via $P(\widehat{\Omega},\theta)=\mathrm{Tr}(\widehat{\Pi}_\theta\,\widehat{\rho})$, where $\widehat{\rho}$ is the density operator that specifies the quantum state. Hence the mean value of the measured outcomes can be defined and then determined as follows:

$$\begin{aligned} \langle\widehat{\Omega}\rangle_Q &= \sum_\theta \theta\, P(\widehat{\Omega},\theta), &\text{(C1)}\\ &= \mathrm{Tr}(\widehat{\Omega}\,\widehat{\rho}). &\text{(C2)} \end{aligned}$$

We can also extend the concept of the mean value for measured outcomes to the case of a non-Hermitian operator $\widehat{\Omega}$—which although it does not correspond to an observable can be written in the form $\widehat{\Omega}=\widehat{\Omega}_1+i\widehat{\Omega}_2$, where both $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ are each observable Hermitian operators, not necessarily commuting. We simple define the mean for $\widehat{\Omega}$ via

$$\begin{aligned} \langle\widehat{\Omega}\rangle &\equiv \langle\widehat{\Omega}_1\rangle+i\langle\widehat{\Omega}_2\rangle\\ &= \mathrm{Tr}[(\widehat{\Omega}_1+i\widehat{\Omega}_2)\widehat{\rho}], \end{aligned} \tag{C3}$$

where $\langle\widehat{\Omega}_1\rangle$ and $\langle\widehat{\Omega}_2\rangle$ are defined as in (C1), and we see that the result is given by the trace process. This definition and result can be applied to provide a meaning for the quantum mean values of operators such as an annihilation operator $\widehat{a}=\frac{1}{\sqrt{2}}(\widehat{x}_A+i\widehat{p}_A)$—which can be written in terms of *quadrature operators* or a transition operator $\widehat{b}^\dagger\widehat{a}=\widehat{S}_x+i\widehat{S}_y$—which can be expressed in terms of *spin operators*. The latter case applies for considering *correlation tests*. If $\widehat{\Omega}$ can be written as the sum of products of Hermitian subsystem operators $\widehat{\Omega}_A$ and $\widehat{\Omega}_B$ the last expression can be used to evaluate the mean value based on the quantum probability distributions for measurements of each $\widehat{\Omega}_A$ and $\widehat{\Omega}_B$.

Note that in expressing $\langle\widehat{\Omega}\rangle$ in terms of $\langle\widehat{\Omega}_1\rangle$ and $\langle\widehat{\Omega}_2\rangle$ we are considering the results of two *independent* sets of measurements, one set for $\widehat{\Omega}_1$ and the other for $\widehat{\Omega}_2$. We do not imply that there is a joint probability $P(\omega_1,\omega_2|\Omega_1,\Omega_2,c)$ for simultaneous outcomes ω_1,ω_2 of a combined measurement of Ω_1,Ω_2 following preparation c. We only require *single* measurement probabilities $P(\omega_1|\Omega_1,c)$ and $P(\omega_2|\Omega_2,c)$ to exist in order to define the mean values via $\langle\widehat{\Omega}_1\rangle=\sum_{\omega_1}\omega_1\,P(\omega_1|\Omega_1,c)$, which corresponds to the set of measurements on $\widehat{\Omega}_1$ *alone*. In von Neumann's proof that hidden variable theories were *inconsistent* with quantum theory, he had evidently used the equivalent of $\langle\widehat{\Omega}\rangle=\sum_{\omega_1}\sum_{\omega_2}(\omega_1+i\omega_2)\,P(\omega_1,\omega_2|\Omega_1,\Omega_2,c)$ based on *one* set of measurements, whereas we use just $\langle\widehat{\Omega}\rangle=\sum_{\omega_1}(\omega_1)\,P(\omega_1|\Omega_1,c)+i\sum_{\omega_2}(\omega_2)\,P(\omega_2|\Omega_2,c)$, which rests on two independent sets of measurements.

In the case of quantum separable states the *mean values* for jointly measuring Ω_A in subsystem A and Ω_B in subsystem B for preparation ρ would be given by

$$\langle \Omega_A \Omega_B \rangle = \sum_R P_R \, \langle \Omega_A \rangle_R \, \langle \Omega_B \rangle_R, \tag{C4}$$

where $\langle \Omega_A \rangle_R = \sum_\alpha \alpha P_Q(\alpha|\Omega_A, \rho, R) = \mathrm{Tr}(\widehat{\Omega}_A \widehat{\rho}_R^A)$ and $\langle \Omega_B(\lambda) \rangle_Q = \sum_\beta \beta P_Q(\beta|\Omega_B, \rho, R) = \mathrm{Tr}(\widehat{\Omega}_B \widehat{\rho}_R^B)$ are the mean values for measurement outcomes for Ω_A and Ω_B. For the quantum separable state the mean value for *any* sum of products of subsystem operators which is Hermitian overall would be given by

$$\left\langle \sum_i \widehat{\Omega}_{Ai} \widehat{\Omega}_{Bi} \right\rangle = \sum_R P_R \sum_i \langle \widehat{\Omega}_{Ai} \rangle_R \, \langle \widehat{\Omega}_{Bi} \rangle_R, \tag{C5}$$

where $\langle \widehat{\Omega}_{Ai} \rangle_R = \mathrm{Tr}(\widehat{\Omega}_{Ai} \widehat{\rho}_R^A)$ and $\langle \widehat{\Omega}_{Bi} \rangle_R = \mathrm{Tr}(\widehat{\Omega}_{Bi} \widehat{\rho}_R^B)$ are quantum mean values, since we can always write $\widehat{\Omega}_{Ai} = \widehat{\Omega}_{Ai}^{(1)} + i\widehat{\Omega}_{Ai}^{(2)}$ where both $\widehat{\Omega}_{Ai}^{(1)}$ and $\widehat{\Omega}_{Ai}^{(2)}$ are Hermitian and can be regarded as observables. So with $\widehat{\Omega}_{Ai}\widehat{\Omega}_{Bi} = \widehat{\Omega}_{Ai}^{(1)}\widehat{\Omega}_{Bi}^{(1)} - \widehat{\Omega}_{Ai}^{(2)}\widehat{\Omega}_{Bi}^{(2)} + i(\widehat{\Omega}_{Ai}^{(1)}\widehat{\Omega}_{Bi}^{(2)} - \widehat{\Omega}_{Ai}^{(2)}\widehat{\Omega}_{Bi}^{(1)})$, which is of the form $\widehat{\Omega}_1 + i\widehat{\Omega}_2$, where both $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ are each observable Hermitian operators (the A and B operators commute), we can then invoke the probability distributions for the $\widehat{\Omega}_{Ai}^{(1)}$, $\widehat{\Omega}_{Bi}^{(1)}$, $\widehat{\Omega}_{Ai}^{(2)}$ and $\widehat{\Omega}_{Bi}^{(2)}$ to derive the expression for the mean value of $\widehat{\Omega}_{Ai}\widehat{\Omega}_{Bi}$ by also using (C3). So (C5) applies even if quantum operators $\widehat{\Omega}_{Ai}$ and $\widehat{\Omega}_{Bi}$ do not represent observables.

Variances can be obtained based on considering the mean values of the square of $\widehat{\Omega}$. For an observable represented by a Hermitian operator $\widehat{\Omega}$ the variance is defined by the mean of the squared variation of outcomes from the mean and equal to the difference between the mean of $\widehat{\Omega}^2$ and the square of the mean of $\widehat{\Omega}$:

$$\begin{aligned}\langle \Delta \widehat{\Omega}^2 \rangle_Q &= \sum_\theta (\theta - \langle \widehat{\Omega} \rangle_Q)^2 P(\widehat{\Omega}, \theta), \\ &= \langle \widehat{\Omega}^2 \rangle_Q - \langle \widehat{\Omega} \rangle_Q^2.\end{aligned} \tag{C6}$$

In the case of a *mixed* state (such as the QSS)

$$\widehat{\rho} = \sum_R P_R \, \widehat{\rho}_R \tag{C7}$$

the *mean* for a Hermitian operator $\widehat{\Omega}$ is the average of means for separate components,

$$\langle \widehat{\Omega} \rangle = \sum_R P_R \, \langle \widehat{\Omega} \rangle_R, \tag{C8}$$

where $\langle \widehat{\Omega} \rangle_R = \mathrm{Tr}(\widehat{\rho}_R \widehat{\Omega})$. The variance for a Hermitian operator $\widehat{\Omega}$ in a mixed state is always never less than the the average of the variances for the separate components (see Ref. [35])

$$\langle \Delta \widehat{\Omega}^{\,2} \rangle \geqslant \sum_R P_R \, \langle \Delta \widehat{\Omega}_R{}^2 \rangle_R, \tag{C9}$$

where $\langle \Delta \widehat{\Omega}^2 \rangle = \mathrm{Tr}(\widehat{\rho} \Delta \widehat{\Omega}^2)$ with $\Delta \widehat{\Omega} = \widehat{\Omega} - \langle \widehat{\Omega} \rangle$, and $\langle \Delta \widehat{\Omega}^2 \rangle_R = \mathrm{Tr}(\widehat{\rho}_R \Delta \widehat{\Omega}_R^2)$ with $\Delta \widehat{\Omega}_R = \widehat{\Omega} - \langle \widehat{\Omega} \rangle_R$. To prove this result we are using (C8) for both $\widehat{\Omega}$ and $\widehat{\Omega}^2$,

$$\begin{aligned}\langle \Delta \widehat{\Omega}^{\,2} \rangle &= \langle \widehat{\Omega}^{\,2} \rangle - \langle \widehat{\Omega} \rangle^2, \\ &= \sum_R P_R \left(\langle \widehat{\Omega}^2 \rangle_R - \langle \widehat{\Omega} \rangle_R^2 \right) \\ &\quad + \sum_R P_R \, \langle \widehat{\Omega} \rangle_R^2 - \left(\sum_R P_R \, \langle \widehat{\Omega} \rangle_R \right)^2, \\ &= \sum_R P_R \left\langle \Delta \widehat{\Omega}_R^2 \right\rangle_R + \sum_R P_R \, \langle \widehat{\Omega} \rangle_R^2 - \left(\sum_R P_R \, |\langle \widehat{\Omega} \rangle_R| \right)^2.\end{aligned} \tag{C10}$$

The variance result (C9) follows because the sum of the last two terms is always $\geqslant 0$ using the result (135) in Appendix E of Ref. [2], with $C_R = \langle \widehat{\Omega} \rangle_R^2$, and $\sqrt{C_R} = |\langle \widehat{\Omega} \rangle_R|$, which are real and positive.

In considering the means and variances in the context of LHVT several difficult issues need to be dealt with. First, in a LHV the observables are basically considered as classical c-numbers, but given that the predictions from quantum theory are accepted as being correct these classical observables must correspond to underlying quantum Hermitian operators—especially as when a LHS occurs where the probabilities $P_Q(\beta|\Omega_B, c, \lambda)$ for subsystem B are also to be given by quantum formulas. Also, there are several entanglement tests involving spin components, these are represented by the spin operators $\widehat{S}_x = (\widehat{b}^\dagger \widehat{a} + \widehat{a}^\dagger \widehat{b})/2$, $\widehat{S}_y = (\widehat{b}^\dagger \widehat{a} - \widehat{a}^\dagger \widehat{b})/2i$ and $\widehat{S}_z = (\widehat{b}^\dagger \widehat{b} - \widehat{a}^\dagger \widehat{a})/2$, where $\widehat{a}$ and $\widehat{b}$ are mode annihilation operators. The tests also involve the total number operator $\widehat{N} = (\widehat{b}^\dagger \widehat{b} + \widehat{a}^\dagger \widehat{a})$. All these operators are Hermitian and represent observable quantities applying for the overall two-mode system. We may also consider number operators for the two modal subsystems defined by $\widehat{N}_A = \widehat{a}^\dagger \widehat{a}$ and $\widehat{N}_B = \widehat{b}^\dagger \widehat{b}$, which again are Hermitian and represent observable quantities for each subsystem. The question then arises: How do you define the spin components and the boson number when the observables are supposed to be nonquantum? Second, when considering entanglement tests involving spin components, both subsystem A and B involve mode annihilation operators—which are non-Hermitian and not themselves associated with measurable observables. What meaning can we give to LHVT probabilities $P(\alpha|\Omega_A, c, \lambda)$ and associated mean values $\langle \Omega_A(\lambda) \rangle = \sum_\alpha \alpha \, P(\alpha|\Omega_A, c, \lambda)$ for subsystem A when during the discussion of spin squeezing tests we consider situations where Ω_A corresponds to a mode annihilation or creation operator? Do we need to consider nonlocal HVT probabilities $P(\alpha_1, \alpha_2|\Omega_{A1}, \Omega_{A2}, c, \lambda)$ associated with the outcomes of measuring *two* observables Ω_{A1}, Ω_{A2} for subsystem A when the hidden variables are λ and which may correspond to quantum operators that do not commute? What happens when we need to consider a product such as $\Omega_{A1}\Omega_{A2}\Omega_{B1}\Omega_{B2}$ such as may occur when we are considering expressions for variances? Would this mean that for products of subsystem observables we should use the expression

$$\langle \Omega_{A1}\Omega_{A2}\Omega_{B1}\Omega_{B2} \rangle = \sum_\lambda P(\lambda|c) \langle \Omega_{A1}\Omega_{A2}(\lambda) \rangle \, \langle \Omega_{B1}\Omega_{B2}(\lambda) \rangle_Q, \tag{C11}$$

where

$$\langle \Omega_{A1}\Omega_{A2}(\lambda)\rangle = \sum_{\alpha_1,\alpha_2} \alpha_{\ 1}\alpha_2\, P(\alpha_1,\alpha_2|\Omega_{A1},\Omega_{A2},c,\lambda),$$
$$\langle \Omega_{B1}\Omega_{B2}(\lambda)\rangle_Q = \sum_{\beta_1,\beta_2} \beta_{\ 1}\beta_2\, P_Q(\beta_1,\beta_2|\Omega_{B1},\Omega_{B2},c,\lambda), \tag{C12}$$

to determine the mean values? But what meaning is there to the quantum expression when the corresponding operators $\widehat{\Omega}_{B1}$, $\widehat{\Omega}_{B2}$ do not commute?

None of these questions arose in considering whether spin squeezing is a test for standard quantum entanglement, since no hidden variables are involved nor are issues of the existence of probabilities for measurement of individual subsystem operators that may become involved in the evaluation. However, when nonquantum LHVT expressions for measurement probabilities are involved, the analogous results to those for quantum mean values need further consideration. Until these issues are resolved we cannot begin to modify the operator-based proof regarding the consequences for spin variances and means for LHVT state. The proof would involve expressions giving meaningful interpretations to the mean values of what would appear to be nonphysical quantities such as mode annihilation and creation operators for subsystem A.

2. General results for mean and variance in LHVT

Before dealing with the above issues it is useful to prove some results for mean values and variances in general HVT that are analogous to similar results in quantum theory. We now consider the measurement of an observable Ω with outcomes ω for a preparation process c. The probability $P(\omega|\Omega,c)$ for this outcome can be written in LHV as

$$P(\omega|\Omega,c) = \sum P(\lambda|c)\, P(\omega|\Omega,c,\lambda), \tag{C13}$$

where λ are the hidden variables and $P(\lambda|c)$ is the probability for preparation process c that the hidden variables are λ and $P(\omega|\Omega,c,\lambda)$ is the probability of outcome ω for measurement of Ω when the hidden variables are λ.

The *mean value* for measurement outcomes for observable Ω will then be given by

$$\langle\Omega\rangle = \sum_{\omega} \omega\, P(\omega|\Omega,c) \tag{C14}$$
$$= \sum_{\lambda} P(\lambda|c)\,\langle\Omega(\lambda)\rangle, \tag{C15}$$
$$\langle\Omega(\lambda)\rangle = \sum_{\omega} \omega\, P(\omega|\Omega,c,\lambda), \tag{C16}$$

where the first equation is the definition and the second equation shows that the mean value is given by weighting the mean value $\langle\Omega(\lambda)\rangle$ that would apply if the hidden variables are λ, by the probability $P(\lambda|c)$ for these hidden variables when the preparation is c. The result (C15) is similar to the quantum result for the mixed state $\widehat{\rho} = \sum_R P_R\widehat{\rho}_R$ where $\langle\widehat{\Omega}\rangle = \sum P_R\langle\widehat{\Omega}\rangle_R$ and $\langle\widehat{\Omega}\rangle_R = \mathrm{Tr}(\widehat{\Omega}\,\widehat{\rho}_R)$. The result for the mean value of a *function* $F(\Omega)$ would be

$$\langle F(\Omega)\rangle = \sum_{\lambda} P(\lambda|c)\,\langle F(\Omega)_\lambda\rangle,$$
$$\langle F(\Omega)_\lambda\rangle = \sum_{\omega} F(\omega)\, P(\omega|\Omega,c,\lambda). \tag{C17}$$

In the case of *two* observables Ω and Λ with outcomes ω and μ, the mean value for a *function* $F(\Omega,\Lambda)$ when the preparation process is c, would be

$$\langle F(\Omega,\Lambda)\rangle = \sum_{\lambda} P(\lambda|c)\,\langle F(\Omega,\Lambda)_\lambda\rangle,$$
$$\langle F(\Omega,\Lambda)_\lambda\rangle = \sum_{\omega\mu} F(\omega,\mu)\, P(\omega,\mu|\Omega,\Lambda,c,\lambda). \tag{C18}$$

This result will be useful when we consider steering tests.

The *variance* for measurement outcomes for observable Ω will then be given by

$$\langle\Delta\Omega^2\rangle = \sum_{\omega} (\omega - \langle\Omega\rangle)^2 P(\omega|\Omega,c), \tag{C19}$$
$$= \sum_{\omega} (\omega^2 - 2\omega\langle\Omega\rangle + \langle\Omega\rangle^2) P(\omega|\Omega,c),$$
$$= \langle\Omega^2\rangle - \langle\Omega\rangle^2, \tag{C20}$$
$$\langle\Omega^2\rangle = \sum_{\omega} \omega^2\, P(\omega|\Omega,c), \tag{C21}$$

where the first equation is the definition and the third equation shows that the variance is given by the difference between the mean of the squared observable and the square of the mean, as in standard statistics. Here we have used $\sum_\omega P(\Omega|\omega,c) = 1$ and (C14). We can then write

$$\langle\Omega^2\rangle = \sum_{\lambda} P(\lambda|c)\,\langle\Omega^2(\lambda)\rangle, \tag{C22}$$
$$\langle\Omega^2(\lambda)\rangle = \sum_{\omega} \omega^2\, P(\omega|\Omega,\lambda,c), \tag{C23}$$

where the second line gives the definition for the mean of the square of the observable when the hidden variables are λ and the first line expresses the mean of the square of the observable in terms of an average over this quantity.

We then have

$$\langle\Delta\Omega^2\rangle = \sum_{\lambda} P(\lambda|c)\,\langle\Omega^2(\lambda)\rangle - \left[\sum_{\lambda} P(\lambda|c)\,\langle\Omega(\lambda)\rangle\right]^2,$$
$$\geqslant \sum_{\lambda} P(\lambda|c)\,[\langle\Omega^2(\lambda)\rangle - \langle\Omega(\lambda)\rangle^2]$$
$$+ \sum_{\lambda} P(\lambda|c)\langle\Omega(\lambda)\rangle^2 - \left[\sum_{\lambda} P(\lambda|c)|\langle\Omega(\lambda)\rangle|\right]^2,$$
$$\geqslant \sum_{\lambda} P(\lambda|c)[\langle\Omega^2(\lambda)\rangle - \langle\Omega(\lambda)\rangle^2], \tag{C24}$$

which establishes an important inequality. The second line follows from the modulus of a sum being less than the sum of the moduli, and the last line follows from the Cauchy inequality $\sum_R P_R C_R \geqslant (\sum_R P_R \sqrt{C_R})^2$ with $\sqrt{C_R} = |\langle\Omega(\lambda)\rangle|$. But we also have

$$\langle\Delta\Omega^2(\lambda)\rangle = \sum_\omega [\omega - \langle\Omega(\lambda)\rangle]^2 P(\omega|\Omega, c, \lambda) \tag{C25}$$

$$= \sum_\omega \omega^2 P(\omega|\Omega, c, \lambda) - \langle\Omega(\lambda)\rangle^2$$

$$= \langle\Omega^2(\lambda)\rangle - \langle\Omega(\lambda)\rangle^2 \tag{C26}$$

showing that when the hidden variable is λ the variance for measured outcomes of observable Ω is equal to the difference between the mean value for measured outcomes of the square of the observable and the square of the mean value (as expected).

We finally have the inequality

$$\langle\Delta\Omega^2\rangle \geqslant \sum_\lambda P(\lambda|c)\,\langle\Delta\Omega^2(\lambda)\rangle. \tag{C27}$$

This result may be compared to the quantum theory result $\langle\Delta\widehat{\Omega}^2\rangle \geqslant \sum_R P_R \langle\Delta\widehat{\Omega}^2\rangle_R$.

Finally, we consider mean values in general HVT for complex combinations of observables Ω_1 and Ω_2, which have measured outcomes ω_1 and ω_2. We can easily show that

$$\langle(\Omega_1 + i\Omega_2)\rangle = \langle\Omega_1\rangle + i\langle\Omega_2\rangle, \tag{C28}$$

where in HVT we have

$$\begin{aligned}\langle\Omega_1\rangle &= \sum_\lambda P(\lambda|c) \sum_{\omega_1} \omega_1 P(\omega_1, \omega_2|\Omega_1, \Omega_2, c, \lambda),\\ \langle\Omega_2\rangle &= \sum_\lambda P(\lambda|c) \sum_{\omega_2} \omega_2 P(\omega_1, \omega_2|\Omega_1, \Omega_2, c, \lambda),\end{aligned} \tag{C29}$$

since the fundamental probability $P(\omega_1, \omega_2|\Omega_1, \Omega_2, c, \lambda)$ always exists in a LHV, *even if* in quantum theory the corresponding operators $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ do *not* commute. This is an important feature to recognize about LHV. The result (C28) may be compared to the quantum result (C3). Thus, we see that many results in HVT are analogous to the results in quantum theory.

With these results now established we can see that for Category 2 states the *mean values* for jointly measuring Ω_A in subsystem A and Ω_B in subsystem B for preparation c would be given by

$$\langle\Omega_A \otimes \Omega_B\rangle = \sum_\lambda P(\lambda|c)\langle\Omega_A(\lambda)\rangle\,\langle\Omega_B(\lambda)\rangle_Q, \tag{C30}$$

where $\langle\Omega_A(\lambda)\rangle = \sum_\alpha \alpha\, P(\alpha|\Omega_A, c, \lambda)$ and $\langle\Omega_B(\lambda)\rangle_Q = \sum_\beta \beta\, P_Q(\beta|\Omega_B, c, \lambda) = \mathrm{Tr}(\widehat{\Omega}_B\, \widehat{\rho}_\lambda^B)$ are the definitions of the mean values for measurement outcomes for Ω_A and Ω_B. The latter is also determined from quantum theory; the former is not. *Variances* can be obtained based on considering the mean values of the squares of Ω_A and Ω_B. The similarities and differences between the Category 2 states and the quantum Separable (Category 1) states expressions (C30) and (C4) should be noted.

3. Links between quantum theory and LHVT

We will also need to consider the mean values for observables which in quantum theory are given by the *sum* of *products* of subsystem Hermitian operators, where the operators for each subsystem do not necessarily commute—$[\widehat{\Omega}_{A1}, \widehat{\Omega}_{A2}] \neq 0$ etc. The links between quantum theory and LHVT for these cases are set out here. Thus for

$$\widehat{\Omega} = \widehat{\Omega}_{A1} \otimes \widehat{\Omega}_{B1} + \widehat{\Omega}_{A2} \otimes \widehat{\Omega}_{B2} \tag{C31}$$

the mean value will be given in *quantum theory* by

$$\begin{aligned}\langle\widehat{\Omega}\rangle &= \langle\widehat{\Omega}_{A1} \otimes \widehat{\Omega}_{B1}\rangle + \langle\widehat{\Omega}_{A2} \otimes \widehat{\Omega}_{B2}\rangle,\\ &= \mathrm{Tr}(\widehat{\Omega}_{A1} \otimes \widehat{\Omega}_{B1})\widehat{\rho} + \mathrm{Tr}(\widehat{\Omega}_{A2} \otimes \widehat{\Omega}_{B2})\widehat{\rho},\\ &= \sum_{\alpha_1\beta_1} \alpha_1\beta_1\, P(\alpha_1, \beta_1|\Omega_{A1}, \Omega_{B1}, c)\\ &\quad + \sum_{\alpha_2\beta 2} \alpha_2\beta_2\, P(\alpha_2, \beta_2|\Omega_{A2}, \Omega_{B2}, c),\end{aligned} \tag{C32}$$

where

$$\begin{aligned}P(\alpha_1, \beta_1|\Omega_{A1}, \Omega_{B1}, c) &= \mathrm{Tr}(\widehat{\Pi}_{\alpha 1} \otimes \widehat{\Pi}_{\beta 1})\widehat{\rho},\\ P(\alpha_2, \beta_2|\Omega_{A2}, \Omega_{B2}, c) &= \mathrm{Tr}(\widehat{\Pi}_{\alpha 2} \otimes \widehat{\Pi}_{\beta 2})\widehat{\rho}.\end{aligned} \tag{C33}$$

In *LHVT* the corresponding observable is

$$\Omega = \Omega_{A1} \otimes \Omega_{B1} + \Omega_{A2} \otimes \Omega_{B2}, \tag{C34}$$

and for *Bell local* states, the mean value of Ω is given by

$$\begin{aligned}\langle\Omega\rangle &= \langle\Omega_{A1} \otimes \Omega_{B1}\rangle + \langle\Omega_{A2} \otimes \Omega_{B2}\rangle,\\ &= \sum_\lambda P(\lambda|c)\,\langle\Omega_{A1}(\lambda)\rangle\,\langle\Omega_{B1}(\lambda)\rangle\\ &\quad + \sum_\lambda P(\lambda|c)\,\langle\Omega_{A2}(\lambda)\rangle\,\langle\Omega_{B2}(\lambda)\rangle,\\ &= \sum_{\alpha_1\beta_1} \alpha_1\beta_1\, P(\alpha_1, \beta_1|\Omega_{A1}, \Omega_{B1}, c)\\ &\quad + \sum_{\alpha_2\beta 2} \alpha_2\beta_2\, P(\alpha_2, \beta_2|\Omega_{A2}, \Omega_{B2}, c),\end{aligned} \tag{C35}$$

where in LHVT

$$\begin{aligned}&P(\alpha_1, \beta_1|\Omega_{A1}, \Omega_{B1}, c)\\ &\quad = \sum_\lambda P(\lambda|c)P(\alpha_1|\Omega_{A1}, c, \lambda)P(\beta_1|\Omega_{B1}, c, \lambda),\\ &P(\alpha_2, \beta_2|\Omega_{A2}, \Omega_{B2}, c)\\ &\quad = \sum_\lambda P(\lambda|c)P(\alpha_2|\Omega_{A2}, c, \lambda)P(\beta_2|\Omega_{B2}, c, \lambda),\end{aligned} \tag{C36}$$

We will use these expressions (C32) and (C35) to interconvert between quantum theory and LHVT when the latter applies.

To determine these mean values *experimentally*, two sets of joint measurements for $\widehat{\Omega}_{A1}$, $\widehat{\Omega}_{B1}$ and *then* $\widehat{\Omega}_{A2}$, $\widehat{\Omega}_{B2}$ (or the classical observables Ω_{A1}, Ω_{B1} and then Ω_{A2}, Ω_{B2}) would be required, unless a technique exists for measuring the outcomes for $\widehat{\Omega}$ (or Ω) directly.

APPENDIX D: CLASSICAL OBSERVABLES AND QUADRATURE AMPLITUDES

For the square of the spin components S_x^2 and S_y^2 we have

$$S_x^2 = \tfrac{1}{4}\left(x_A^2 x_B^2 + p_A^2 p_B^2\right) + \tfrac{1}{2}(U_A U_B - V_A V_B), \tag{D1}$$

$$S_y^2 = \tfrac{1}{4}\left(p_A^2 x_B^2 + x_A^2 p_B^2\right) - \tfrac{1}{2}(U_A U_B + V_A V_B), \tag{D2}$$

and the square of $X_\theta(\pm)$ is given by

$$\begin{aligned} X_\theta(\pm)^2 = &\tfrac{1}{2}\left(x_A^2\cos^2\theta + p_A^2\sin^2\theta + 2U_A\sin\theta\cos\theta\right)\\ &+\tfrac{1}{2}\left(x_B^2\cos^2\theta + p_B^2\sin^2\theta + 2U_B\sin\theta\cos\theta\right)\\ &\pm(x_A x_B\cos^2\theta + p_A p_B\sin^2\theta\\ &+ x_A p_B\sin\theta\cos\theta + p_A x_B\sin\theta\cos\theta). \end{aligned} \tag{D3}$$

APPENDIX E: WERNER STATES

As examples of the three categories of Bell local states we may consider the states introduced by Werner [12] as $U\otimes U$ invariant states $((\widehat{U}\otimes\widehat{U})\widehat{\rho}_W(\widehat{U}^\dagger\otimes\widehat{U}^\dagger) = \widehat{\rho}_W$, where $\widehat{U}$ is any *unitary* operator) for two d-dimensional subsystems. Depending on the parameter η (or ϕ) the Werner states, may be separable or entangled. They may also be Bell local in one of the three categories described above, or they may be Bell nonlocal. The density operator for the *Werner states* is given by

$$\begin{aligned} \widehat{\rho}_W &= (d^3-d)^{-1}[(d-\phi)\widehat{1} + (d\phi - 1)\widehat{V}]\\ &= \left[\frac{(d-1+\eta)}{(d-1)}\right]\frac{\widehat{1}}{d^2} - \left[\frac{\eta}{(d-1)}\right]\frac{\widehat{V}}{d}, \end{aligned} \tag{E1}$$

where $\widehat{1}$ is the *unit* operator and $\widehat{V}$ is the *flip* operator defined as $\widehat{V}(|\psi\rangle\otimes|\chi\rangle) = |\chi\rangle\otimes|\psi\rangle$. The two expressions are interconvertable with $\phi = [1-(d+1)\eta]/d$. For a positive density operator we have $-1 \leqslant \phi \leqslant +1$. Werner has shown that if $\eta < 1/(d+1)$ (or $\phi > 0$) the state $\widehat{\rho}_W$ is separable, but for $\eta > 1/(d+1)$ (or $\phi < 0$) the state is entangled. Thus Werner states with $\eta < 1/(d+1)$ or $\phi > 0$ are separable. Wiseman *et al.* [6] considered the above categories for such Werner states and determined the parameter boundaries for the various categories. These results are shown in Fig. 2 (taken from Fig. 1(a) in Ref. [6]), where the parameter regimes for the various categories of quantum states are explained.

APPENDIX F: IDEA OF EPR STEERING

In this Appendix we consider for reasons of completeness the physical idea behind EPR steering, as presented in the papers [6–8].

We can derive expressions within LHV theory for the *conditional probabilities* defined in (B5). These expressions apply for all three Bell local categories considered here. We will focus on LHSs, which in terms of our LHVCS may be either in Category 1 or Category 2. We will initially consider the latter.

In the case of *Category 2* states (which are *LHSs*) we obtain from (27) and (B5)

$$P(\beta|\Omega_B||\alpha,\Omega_A,c) = \frac{\sum_\lambda P(\alpha|\Omega_A,c,\lambda)\,\mathrm{Tr}_B((\widehat{\Pi}^B_\beta)\widehat{\rho}^B(\lambda))}{\sum_\lambda P(\alpha|\Omega_A,c,\lambda)\,P(\lambda|c)} \tag{F1}$$

using (11) and (29).

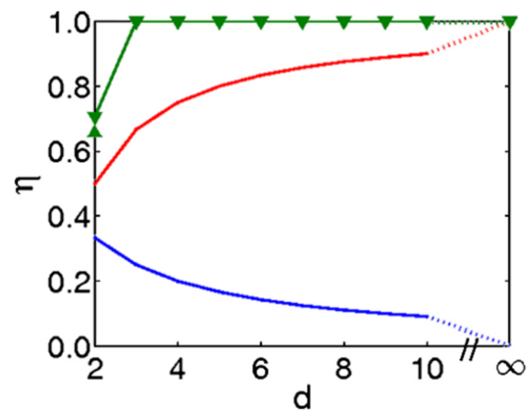

FIG. 2. Parameter η (see text) boundaries for Werner states. The blue line corresponds to $\eta = 1/(d+1)$, the red line to $\eta = (1-d^{-1})$ and the green line to $\eta = 1$ for $d \geqslant 3$. For η below blue line the states are Category 1-separable states. These states are also Bell local, LHS, and nonsteerable. For η between blue line and red line the states are Category 2. These states are also Bell local, nonsteerable, and entangled. For η between red line and green line the states are Category 3—Bell local, steerable, and entangled (EPR entangled). For η above green line the states are Category 4—Bell nonlocal, steerable, and entangled. This is possible only for $d = 2$. Figure taken from Wiseman *et al.* Ref. [6].

It is also important to realize that these model LHSs are still related to an overall quantum state, but one which is *nonseparable* since we cannot derive the density operator (20) for separable states from Category 2 expression (27) for the joint probability. For Category 2 LHSs, $P(\alpha|\Omega_A,c,\lambda)$ is *not* given by a quantum expression. However, as in Refs. [7,8] we can relate the quantities in the LHS model (27) to a density operator for subsystem B that is *conditional* on the results for measurements on subsystem A.

From (2) the *quantum theory* result for the probability that measurement of observable Ω_A results in outcome α is given by

$$P(\alpha|\Omega_A,\rho) = \mathrm{Tr}\left(\left(\widehat{\Pi}^A_\alpha\otimes\widehat{1}^B\right)\widehat{\rho}\right), \tag{F2}$$

where $\widehat{\rho}$ is the density operator for the overall quantum state (the preparation symbol c is left out for simplicity). In the Copenhagen interpretation of quantum theory the *normalized* state that is produced as a *result* of this measurement is the *conditional state*

$$\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho) = \left(\widehat{\Pi}^A_\alpha\otimes\widehat{1}^B\right)\widehat{\rho}\left(\widehat{\Pi}^A_\alpha\otimes\widehat{1}^B\right)/P(\alpha|\Omega_A,\rho). \tag{F3}$$

This state has a trace of unity, as required. To confirm that $\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho)$ *does* lead to the correct quantum expression for the *conditional probability* $P(\beta|\Omega_B||\alpha|\Omega_A,\rho)$ (i.e., that measurement of Ω_B in subsystem B will result in outcome β *given* that measurement of Ω_A resulted in outcome α based on the quantum state $\widehat{\rho}$), we calculate the probability of that measurement of Ω_B in subsystem B which will result in outcome β for the quantum state $\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho)$. This is given by

$$\begin{aligned}
P(\beta|\Omega_B,\rho_{\mathrm{cond}}) &= \mathrm{Tr}\big((\widehat{1}^A\otimes\widehat{\Pi}^B_\beta)\,\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho)\big),\\
&= \mathrm{Tr}\big((\widehat{\Pi}^A_\alpha\otimes\widehat{\Pi}^B_\beta)\widehat{\rho}(\widehat{\Pi}^A_\alpha\otimes\widehat{1}^B)\big)/P(\alpha|\Omega_A,\rho),\\
&= \mathrm{Tr}\big((\widehat{\Pi}^A_\alpha\otimes\widehat{\Pi}^B_\beta)\,\widehat{\rho}\big)/P(\alpha|\Omega_A,\rho),\\
&= P(\alpha,\beta|\Omega_A,\Omega_B,\rho)/P(\alpha|\Omega_A,\rho),\\
&= P(\beta|\Omega_B\,||\alpha|\Omega_A,\rho),
\end{aligned}\tag{F4}$$

using the cyclic properties of the trace and $(\widehat{\Pi}^A_\alpha)^2=\widehat{\Pi}^A_\alpha$, with the last line [see (B5)] following from *Bayes' theorem*. This confirms the status of $\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho)$.

The *physical* concept of *steering* has been discussed in several papers, including Refs. [6,7] and [8] and was originally introduced by Schrödinger [10] following the important EPR paper [5]. The key idea is that when a measurement of Ω_A is made on subsystem A resulting in outcome α (the bipartite quantum state prepared being ρ) this results in both the overall quantum state changing to a new conditioned state $\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho)$ [given in Eq. (F3)] and hence the *postmeasurement* state describing subsystem B changing to

$$\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho)^B=\mathrm{Tr}_A(\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho))\tag{F5}$$

from its *premeasurement* state $\widehat{\rho}^B=\mathrm{Tr}_A(\widehat{\rho})$ given by the *reduced density operator* (Eq. 3). This strange quantum effect allows for an experiment carried out on subsystem A to instantly change (or "*steer*") the quantum state for subsystem B into a new quantum state, even when the two subsystems are localized in well-separated spatial regions and the experimenter on A may have no direct access to subsystem B. For those who accept the Copenhagen interpretation of quantum theory there is nothing really strange involved. Quantum states merely specify all that can be known about the physical state (and no distinction between "physical state" and "quantum state" is made), so as the measurement of Ω_A has led to a particular outcome α our knowledge about the state has changed, and hence the quantum state for both the overall system and its subsystems should change accordingly. Using quantum theory we can obtain an explicit formula for $\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho)^B$, and this is

$$\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho)^B=\sum_{\beta l,\gamma n}|B\beta l\rangle\langle B\gamma n|\sum_i\rho_{A\alpha i,B\beta l::A\alpha i,B\gamma n},\tag{F6}$$

where the original density operator ρ is expressed in terms of orthonormal basis states $|A\alpha i\rangle\otimes|B\beta n\rangle$ that are eigenstates for $\widehat{\Omega}_A$ and $\widehat{\Omega}_B$, with $i=1,2,\dots,d_\alpha$ and $n=1,2,\dots,d_\beta$ allowing for degeneracy.

We can also show that the sum of the conditional density operators $\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho)^B$ each weighted by the probability $P(\alpha|\Omega_A,\rho)$ for the measurement outcome α for Ω_A gives the reduced density operator $\widehat{\rho}^B$ associated with the original state ρ. This result is not surprising, since carrying out the measurement of any choice of Ω_A and then discarding the results would be described by reduced density operator:

$$\sum_\alpha P(\alpha|\Omega_A,\rho)\,\widehat{\rho}_{\mathrm{cond}}(\alpha|\Omega_A,\rho)^B=\widehat{\rho}^B=\mathrm{Tr}_A\widehat{\rho}.\tag{F7}$$

The proofs of (F6) and (F7) are straightforward.

Thus, we have seen how according to quantum theory the quantum state describing subsystem B changes as a result of measuring Ω_A on subsystem A and obtaining outcome α. Furthermore, we have obtained quantum theory expressions (F4) for the conditional probability $P(\beta|\Omega_B,\rho_{\mathrm{cond}})$ for measurement of Ω_B on subsystem B and obtaining outcome β when measurement of Ω_A on subsystem A resulted in outcome α and (F6) for the quantum state describing subsystem B. The question then is: Although quantum theory gives the correct results for the conditional probability $P(\beta|\Omega_B,\rho_{\mathrm{cond}})$, can the same results *also* be explained in a local hidden variable theory?

Following the operational definition for steering in Refs. [6–8], the quantum state ρ is only considered to be *EPR steerable* when the conditional probability $P(\beta|\Omega_B||\alpha,\Omega_A,c)$ can *not* be explained via a local hidden variable theory. For the LHS cases of Category 1 and Category 2 states we will see that a LHV theory explanation applies. We consider what expression for a density operator for subsystem B would give the LHS result for the conditional probability $P(\beta|\Omega_B||\alpha,\Omega_A,c)$ for measurement of Ω_B to have outcome β, given that measurement of Ω_A has outcome α and the preparation process is c. In the case of Category 2 states we use Eqs. (27) and (29) in conjunction with (B5) and (11) to find

$$P(\beta|\Omega_B||\alpha,\Omega_A,c)=\frac{\sum_\lambda P(\alpha|\Omega_A,c,\lambda)\,\mathrm{Tr}_B\big((\widehat{\Pi}^B_\beta)\widehat{\rho}^B(\lambda)\big)\,P(\lambda|c)}{\sum_\lambda P(\alpha|\Omega_A,c,\lambda)\,P(\lambda|c)}.\tag{F8}$$

We then *define* a *new* normalized quantum state for subsystem B, $\widehat{\rho}^B_{\mathrm{cond}}(\alpha|\Omega_A,c)$, by the expression

$$\begin{aligned}
\widehat{\rho}^B_{\mathrm{cond}}(\alpha|\Omega_A,c)&=\frac{\sum_\lambda P(\alpha|\Omega_A,c,\lambda)\,\widehat{\rho}^B(\lambda)\,P(\lambda|c)}{\mathrm{Tr}_B\big[\sum_\lambda P(\alpha|\Omega_A,c,\lambda)\,\widehat{\rho}^B(\lambda)\,P(\lambda|c)\big]},\\
&=\frac{\sum_\lambda P(\alpha|\Omega_A,c,\lambda)\,\widehat{\rho}^B(\lambda)\,P(\lambda|c)}{\big[\sum_\lambda P(\alpha|\Omega_A,c,\lambda)\,P(\lambda|c)\big]}.
\end{aligned}\tag{F9}$$

It is to be noted that this state for subsystem B involves local HVT and not quantum expressions for the measurement probabilities $P(\alpha|\Omega_A,c,\lambda)$ for subsystem A. We then see from (2) that for *this state* the probability for measurement of Ω_B to have outcome β is given by

$$\begin{aligned}
&\mathrm{Tr}_B\big(\widehat{\Pi}^B_\beta\,\widehat{\rho}^B_{\mathrm{cond}}(\alpha|\Omega_A,c)\big)\\
&\quad=\frac{\sum_\lambda P(\alpha|\Omega_A,c,\lambda)\,\mathrm{Tr}_B\big((\widehat{\Pi}^B_\beta)\widehat{\rho}^B(\lambda)\big)\,P(\lambda|c)}{\sum_\lambda P(\alpha|\Omega_A,c,\lambda)\,P(\lambda|c)},\\
&\quad=P(\beta|\Omega_B||\alpha,\Omega_A,c),
\end{aligned}\tag{F10}$$

which is the same as (F1) obtained for the Category 2 states (which are LHSs). Thus the subsystem B quantum state (F9) has been constructed purely from the Category 2 LHS model probabilities $P(\alpha|\Omega_A,c,\lambda)$ and $P(\lambda|c)$, together with the model quantum LHS $\widehat{\rho}^B(\lambda)$—which is a possible quantum state for subsystem B based on hidden variables λ. The subsystem B quantum state $\widehat{\rho}^B_{\mathrm{cond}}(\alpha|\Omega_A,c)$ in (F9) determines the correct probability for measurement of Ω_B to have outcome β. The same analysis would apply to the LHSs in Category 1, the only difference being that $P(\alpha|\Omega_A,c,\lambda)$ would be replaced by $P_Q(\alpha|\Omega_A,c,\lambda)$ in terms of our notation. So in both of these cases there could be a hidden state $\widehat{\rho}^B(\lambda)$ associated with hidden variables that could explain [along with suitable choices for $P(\alpha|\Omega_A,c,\lambda)$ and $P(\lambda|c)$] the measurements on subsystem B. The treatment, however, does not apply to the quantum states in Category 3, where the LHV model in Eq. (28) does *not* include a quantum state $\widehat{\rho}^B(\lambda)$ for subsystem B. Hence, the conditional probability $P(\beta|\Omega_B||\alpha,\Omega_A,c)$ *can* be explained via the *LHS model* for both Category 1 and Category 2 states, showing that the Category 1 and Category 2 quantum states are *nonsteerable*. However, the Category 3 states are *EPR steerable*.

APPENDIX G: SPIN VARIANCES: EPR STEERING TEST

The EPR steering test in (73) can be obtained from the results in Secs. IV E and IV F by using (61), (45), and (54). We find using LHVT that for Category 2 states

$$\langle\Delta S_x^2\rangle+\langle\Delta S_y^2\rangle-\tfrac{1}{4}\langle N\rangle+\tfrac{1}{2}\langle S_z\rangle \geqslant \langle N_A\otimes N_B\rangle+\tfrac{1}{2}\langle 1_A\otimes N_B\rangle, \geqslant 0. \tag{G1}$$

Details are as follows:

$$\begin{aligned}&\langle\Delta S_x^2\rangle+\langle\Delta S_y^2\rangle-\tfrac{1}{4}\langle N\rangle+\tfrac{1}{2}\langle S_z\rangle\\ &\quad\geqslant \langle N_A\otimes N_B\rangle+\tfrac{1}{2}\langle 1_A\otimes N_B\rangle+\tfrac{1}{2}\langle N_A\otimes 1_B\rangle-\tfrac{1}{4}\langle 1_A\otimes N_B\rangle\\ &\qquad-\tfrac{1}{4}\langle N_A\otimes 1_B\rangle+\tfrac{1}{4}\langle 1_A\otimes N_B\rangle-\tfrac{1}{4}\langle N_A\otimes 1_B\rangle,\\ &\quad\geqslant \langle N_A\otimes N_B\rangle+\tfrac{1}{2}\langle 1_A\otimes N_B\rangle,\\ &\quad\geqslant 0.\end{aligned} \tag{G2}$$

As LHVT is required to predict the same result as quantum theory, we have

$$\langle\Delta \widehat{S}_x^2\rangle+\langle\Delta \widehat{S}_y^2\rangle-\tfrac{1}{4}\langle\widehat{N}\rangle+\tfrac{1}{2}\langle\widehat{S}_z\rangle \geqslant \langle\widehat{N}_A\otimes\widehat{N}_B\rangle+\tfrac{1}{2}\langle\widehat{1}_A\otimes\widehat{N}_B\rangle, \geqslant 0, \tag{G3}$$

since both $\langle\widehat{N}_A\otimes\widehat{N}_B\rangle$ and $\langle\widehat{1}_A\otimes\widehat{N}_B\rangle$ are positive quantities. In this form it shows that if $\langle\Delta\widehat{S}_x^2\rangle+\langle\Delta\widehat{S}_y^2\rangle-\frac{1}{4}\langle\widehat{N}\rangle+\frac{1}{2}\langle\widehat{S}_z\rangle<0$, then the state cannot be Category 2. This result is also obtained by Appendices I and J.

APPENDIX H: VARIANCES OF TWO-MODE QUADRATURES—CATEGORY 2 STATES

Using the LHVT expressions (46) and (D3) for $X_\theta(\pm)$ and $X_\theta(\pm)^2$ together with the results (49) and (58) for $\langle x_B\rangle$, $\langle p_B\rangle$ and $\langle U_B\rangle$, together with $U_A=\frac{1}{2}(x_Ap_A+p_Ax_A)$, we find for Category 2 states the mean values of the two-mode quadratures and their square are given by

$$\langle X_\theta(\pm)\rangle=\frac{1}{\sqrt{2}}(\langle x_A\rangle\cos\theta+\langle p_A\rangle\sin\theta), \tag{H1}$$

$$\begin{aligned}\langle X_\theta(\pm)\rangle^2&=\frac{1}{2}(\langle x_A\rangle^2\cos^2\theta+\langle p_A\rangle^2\sin^2\theta\\ &\quad+2\langle x_A\rangle\langle p_A\rangle\sin\theta\cos\theta),\\ \langle X_\theta(\pm)^2\rangle&=\frac{1}{2}\big(\langle x_A^2\rangle\cos^2\theta+\langle(x_Ap_A+p_Ax_A)\rangle\sin\theta\cos\theta\\ &\quad+\langle p_A^2\rangle\sin^2\theta\big)+\frac{1}{2}\left(\langle N_B\rangle+\frac{1}{2}\right).\end{aligned} \tag{H2}$$

The variance for Category 2 states is then given by the LHVT expression

$$\begin{aligned}\langle\Delta X_\theta(\pm)^2\rangle&=\tfrac{1}{2}\langle(\Delta x_A\cos\theta+\Delta p_A\sin\theta)\\ &\quad\times(\Delta x_A\cos\theta+\Delta p_A\sin\theta)\rangle\\ &\quad+\tfrac{1}{2}\big(\langle N_B\rangle+\tfrac{1}{2}\big),\\ \langle\Delta P_\theta(\pm)^2\rangle&=\tfrac{1}{2}\langle(-\Delta x_A\sin\theta+\Delta p_A\cos\theta)\\ &\quad\times(-\Delta x_A\sin\theta+\Delta p_A\cos\theta)\rangle\\ &\quad+\tfrac{1}{2}\big(\langle N_B\rangle+\tfrac{1}{2}\big),\end{aligned} \tag{H3}$$

where $\Delta x_A=x_A-\langle x_A\rangle$ and $\Delta p_A=p_A-\langle p_A\rangle$. The expression for $\langle\Delta P_\theta(\pm)^2\rangle$ is obtained using $P_\theta(\pm)=X_{\theta+\pi/2}(\pm)$.

As LHVT underlies quantum theory then we also have for the quantum theory treatment of Category 2 states

$$\begin{aligned}\langle\Delta\widehat{X}_\theta(\pm)^2\rangle&=\tfrac{1}{2}\langle(\Delta\widehat{x}_A\cos\theta+\Delta\widehat{p}_A\sin\theta)\\ &\quad\times(\Delta\widehat{x}_A\cos\theta+\Delta\widehat{p}_A\sin\theta)\rangle\\ &\quad+\tfrac{1}{2}\big(\langle\widehat{1}_A\otimes\widehat{N}_B\rangle+\tfrac{1}{2}\big),\end{aligned} \tag{H4}$$

where now $\Delta\widehat{x}_A=\widehat{x}_A-\langle\widehat{x}_A\rangle$, $\Delta\widehat{p}_A=\widehat{p}_A-\langle\widehat{p}_A\rangle$. However, we can make use of the SSR to simplify these expressions further. As shown in Sec. II A the reduced density operator for subsystem A satisfies the local particle number SSR. This is the case even though the reduced density operator depends on the full density matrix for both subsystems, unlike that for a LHS. Consequently

$$\langle\widehat{x}_A\rangle=\mathrm{Tr}_A(\widehat{x}_A\widehat{\rho}^A)=0,\quad\langle\widehat{p}_A\rangle=\mathrm{Tr}_A(\widehat{p}_A\widehat{\rho}^A)=0, \tag{H5}$$

using the same arguments as for $\langle x_B(\lambda)\rangle_Q$ and $\langle p_B(\lambda)\rangle_Q$ in Eq. (49). Furthermore, the same steps as for $\langle x_B^2(\lambda)\rangle_Q$, $\langle p_B^2(\lambda)\rangle_Q$ and $\langle U_B(\lambda)\rangle_Q$ lead to

$$\begin{aligned}&\langle\widehat{x}_A^2\rangle=\langle\widehat{N}_A\rangle+\tfrac{1}{2},\quad\langle\widehat{p}_A^2\rangle=\langle\widehat{N}_A\rangle+\tfrac{1}{2}\\ &\langle\widehat{U}_A\rangle=0,\end{aligned} \tag{H6}$$

(see Sec. IV E 3). Using these results we then find that

$$\begin{aligned}\langle\Delta\widehat{X}_\theta(\pm)^2\rangle&=\tfrac{1}{2}\big(\langle\widehat{N}_A\otimes\widehat{1}_B\rangle+\tfrac{1}{2}\big)\\ &\quad+\tfrac{1}{2}\big(\langle\widehat{1}_A\otimes\widehat{N}_B\rangle+\tfrac{1}{2}\big),\\ &=\tfrac{1}{2}\langle\widehat{N}\rangle+\tfrac{1}{2},\\ \langle\Delta\widehat{P}_\theta(\pm)^2\rangle&=\tfrac{1}{2}\langle\widehat{N}\rangle+\tfrac{1}{2}.\end{aligned} \tag{H7}$$

[The calculation for $\langle\Delta\widehat{P}_\theta(\pm)^2\rangle$ is trivial, as $\widehat{P}_\theta(\pm)=\widehat{X}_{\theta+\pi/2}(\pm)$]. Exactly the same results apply for Category 1 (separable) states (see Appendix L in Ref. [3]).

APPENDIX I: CORRELATION TESTS FOR EPR STEERING

The paper by Cavalcanti *et al.* [16] derives certain inequalities for $|\langle\widehat{a}^\dagger\widehat{b}\rangle|^2$ for Category 1 and Category 2 states which lead to *strong correlation* tests for EPR steering. We will show here that these inequalities lead to more useful tests in terms of spin operators for quantum entanglement and EPR steering. These inequalities are set out here in Eqs. (I9) and (I11) for Category 1 and Category 2 states, respectively. The inequality in Eq. (I9) has also been previously obtained for separable states by Hillery and Zubairy [14]. They two inequalities correspond to Eqs. (15) and (14) in Ref. [16] where there are $N = 2$ subsystems ("sites"), with Eq. (15) applying when both subsystems are associated with a LHS ($T = 2$—two "trusted sites") and Eq. (14) when only one subsystem has a LHS ($T = 1$—one "trusted site"). The inequalities obtained by Cavalcanti *et al.* [16] were based on their general expression in Eq. (4) for the LHV theory joint measurement probability, for which Eqs. (26) and (27) for Category 1 and Category 2 states are special cases. Hence these inequalities would apply for the present paper. For completeness however, rather than just quoting the inequalities in Ref. [16] we will also derive them here using the approach set out in the present paper. A further inequality for $|\langle\widehat{a}^\dagger\widehat{b}\rangle|^2$ will also be derived that would apply to Category 3 states.

For Category 1 states the result gives a strong correlation test and the Hillery-Zubairy [14] test for quantum entanglement, while for Category 2 states the result gives a strong correlation test plus a generalized Hillery-Zubairy test for EPR steering, originally set out in He *et al.* [15] for the case where $\langle\widehat{S}_z\rangle = 0$. The new test allows for $\langle\widehat{S}_z\rangle \neq 0$. For Category 3 states no useful test for Bell nonlocality occurs.

1. General correlation inequality for $|\langle\widehat{a}^\dagger\widehat{b}\rangle|^2$—Bell local states

Using Eqs. (33) and (44) to introduce quadrature operators and spin operators, the quantity $\widehat{a}^\dagger\widehat{b}$ can be written as

$$\begin{aligned}\widehat{a}^\dagger\widehat{b} &= \tfrac{1}{2}(\widehat{x}_A - i\widehat{p}_A)(\widehat{x}_B + i\widehat{p}_B)\\ &= \widehat{S}_x - i\widehat{S}_y\end{aligned} \tag{I1}$$

so that the LHVT quantity $\langle a^\dagger b\rangle$ becomes

$$\left\langle a^\dagger b\right\rangle = \tfrac{1}{2}(\langle x_A x_B\rangle + \langle p_A p_B\rangle + i(\langle x_A p_B\rangle - \langle p_A x_B\rangle)). \tag{I2}$$

Then introducing the LHVT expression

$$\begin{aligned}\langle a^\dagger b\rangle = \frac{1}{2}\sum_\lambda P(\lambda|c)[\langle x_A(\lambda)\rangle - i\langle p_A(\lambda)\rangle]\\ \times[\langle x_B(\lambda)\rangle + i\langle p_B(\lambda)\rangle]\end{aligned}$$

and

$$\begin{aligned}|\langle a^\dagger b\rangle| \leqslant \frac{1}{2}\sum_\lambda P(\lambda|c)|[\langle x_A(\lambda)\rangle - i\langle p_A(\lambda)\rangle]|\\ \times|[\langle x_B(\lambda)\rangle + i\langle p_B(\lambda)\rangle]|\end{aligned}$$

with $|[\langle x_A(\lambda)\rangle - i\langle p_A(\lambda)\rangle]| = \sqrt{\langle x_A(\lambda)\rangle^2 + \langle p_A(\lambda)\rangle^2}$ etc., we then find that

$$\begin{aligned}|\langle a^\dagger b\rangle|^2 \leqslant \frac{1}{4}\Bigg[\sum_\lambda P(\lambda|c)\sqrt{\langle x_A(\lambda)\rangle^2 + \langle p_A(\lambda)\rangle^2}\\ \times\sqrt{\langle x_B(\lambda)\rangle^2 + \langle p_B(\lambda)\rangle^2}\Bigg]^2.\end{aligned} \tag{I3}$$

Using the inequality (32) with

$$C(\lambda) = [\langle x_A(\lambda)\rangle^2 + \langle p_A(\lambda)\rangle^2][\langle x_B(\lambda)\rangle^2 + \langle p_B(\lambda)\rangle^2] \geqslant 0,$$

we then have the key inequality

$$\begin{aligned}|\langle a^\dagger b\rangle|^2 \leqslant \frac{1}{4}\sum_\lambda P(\lambda|c)[\langle x_A(\lambda)\rangle^2 + \langle p_A(\lambda)\rangle^2]\\ \times[\langle x_B(\lambda)\rangle^2 + \langle p_B(\lambda)\rangle^2]\end{aligned} \tag{I4}$$

that would follow from the approach in Ref. [16]. Again, as LHVT underlies quantum theory we can use (45), (60), (18), and (19) to write this inequality for *all* Bell local states in terms of quantum operators as

$$\begin{aligned}|\langle\widehat{a}^\dagger\widehat{b}\rangle|^2 &\leqslant \langle(\widehat{N}_A + \widehat{V}_A)\otimes(\widehat{N}_B + \widehat{V}_B)\rangle,\\ &= \langle\widehat{N}_A\otimes\widehat{N}_B\rangle + \tfrac{1}{2}\langle\widehat{1}_A\otimes\widehat{N}_B\rangle + \tfrac{1}{2}\langle\widehat{N}_A\otimes\widehat{1}_B\rangle + \tfrac{1}{4}.\end{aligned} \tag{I5}$$

2. Stronger correlation inequalities for Bell local states

Stronger inequalities can now be derived for the quantities $\langle x_A(\lambda)\rangle^2 + \langle p_A(\lambda)\rangle^2$ and $\langle x_B(\lambda)\rangle^2 + \langle p_B(\lambda)\rangle^2$ in the cases of Categories 1, 2, and 3 states. This leads to some outcomes different from (I5).

Even if the subsystem C does *not* involve a LHS $\widehat{\rho}_\lambda^C$ then we can always use the inequality (31) to give $\langle x_C(\lambda)\rangle^2 \leqslant \langle x_C^2(\lambda)\rangle$ and $\langle p_C(\lambda)\rangle^2 \leqslant \langle p_C^2(\lambda)\rangle$. This is equivalent to the variances of x_C and p_C being non-negative. Thus

$$\langle x_C(\lambda)\rangle^2 + \langle p_C(\lambda)\rangle^2 \leqslant \left\langle x_C^2(\lambda)\right\rangle + \left\langle p_C^2(\lambda)\right\rangle. \tag{I6}$$

On the other hand, if the subsystem C *does* involve a LHS $\widehat{\rho}_\lambda^C$, then we can obtain a *stronger inequality* via quantum theory. For any real η the quantity $\langle(\Delta\widehat{x}_C - i\eta\Delta\widehat{p}_C)(\Delta\widehat{x}_C + i\eta\Delta\widehat{p}_C)\rangle_\lambda = \mathrm{Tr}[(\Delta\widehat{x}_C - i\eta\Delta\widehat{p}_C)(\Delta\widehat{x}_C + i\eta\Delta\widehat{p}_C)\widehat{\rho}_\lambda^C] \geqslant 0$, where $\Delta\widehat{x}_C = \widehat{x}_C - \langle\widehat{x}_C\rangle_\lambda$, $\Delta\widehat{p}_C = \widehat{p}_C - \langle\widehat{p}_C\rangle_\lambda$. Thus for all η we have $\langle\Delta\widehat{x}_C^2\rangle_\lambda - \eta + \eta^2\langle\Delta\widehat{p}_C^2\rangle_\lambda \geqslant 0$ using $[\widehat{x}_C, \widehat{p}_C] = i$. Putting $\eta = 1$ gives the inequality $\langle\Delta\widehat{x}_C^2\rangle_\lambda + \langle\Delta\widehat{p}_C^2\rangle_\lambda - 1 \geqslant 0$, which can be written as $\langle\widehat{x}_C\rangle_\lambda^2 + \langle\widehat{p}_C\rangle_\lambda^2 \leqslant \langle\widehat{x}_C^2\rangle_\lambda + \langle\widehat{p}_C^2\rangle_\lambda - 1$. In terms of LHVT notation this inequality is

$$\langle x_C(\lambda)\rangle^2 + \langle p_C(\lambda)\rangle^2 \leqslant \left\langle x_C^2(\lambda)\right\rangle + \left\langle p_C^2(\lambda)\right\rangle - 1. \tag{I7}$$

For Category 1 states both subsystems involve a LHS, so the key inequality (I4) gives

$$\begin{aligned}|\langle a^\dagger b\rangle|^2 \leqslant \frac{1}{4}\sum_\lambda P(\lambda|c)\left[\left\langle x_A^2(\lambda)\right\rangle + \left\langle p_A^2(\lambda)\right\rangle - 1\right]\\ \times\left[\left\langle x_B^2(\lambda)\right\rangle + \left\langle p_B^2(\lambda)\right\rangle - 1\right].\end{aligned} \tag{I8}$$

Using (18), (19), (36), and (35) we can then convert these inequalities to quantum expressions involving number operators, $\widehat{N}_C = \widehat{c}^\dagger\widehat{c}$ (where $C = A, B$):

$$\begin{aligned}|\langle\widehat{a}^\dagger\widehat{b}\rangle|^2 &\leqslant \langle(\widehat{N}_A + \widehat{V}_A - \widehat{1}_A/2)\otimes(\widehat{N}_B + \widehat{V}_B - \widehat{1}_B/2)\rangle,\\ &= \langle\widehat{N}_A\otimes\widehat{N}_B\rangle.\end{aligned} \tag{I9}$$

For Category 2 states with subsystem B involving a LHS $\widehat{\rho}_\lambda^B$, the key inequality (I4) gives

$$\begin{aligned}|\langle a^\dagger b\rangle|^2 \leqslant \frac{1}{4}\sum_\lambda P(\lambda|c)\left[\left\langle x_A^2(\lambda)\right\rangle + \left\langle p_A^2(\lambda)\right\rangle\right]\\ \times\left[\left\langle x_B^2(\lambda)\right\rangle + \left\langle p_B^2(\lambda)\right\rangle - 1\right].\end{aligned} \tag{I10}$$

Similarly to the Category 1 case we then find that for Category 2 states (with B involving the LHS)

$$\begin{aligned}|\langle\widehat{a}^\dagger\widehat{b}\rangle|^2 &\leqslant \left\langle(\widehat{N}_A + \widehat{V}_A)\otimes\left(\widehat{N}_B + \widehat{V}_B - \tfrac{1}{2}\widehat{1}_B\right)\right\rangle\\ &= \left\langle\left(\widehat{N}_A + \tfrac{1}{2}\widehat{1}_A\right)\otimes\widehat{N}_B\right\rangle.\end{aligned} \tag{I11}$$

For Category 3 states with neither subsystem involving a LHS, the key inequality (I4) gives

$$|\langle a^{\dagger}b\rangle|^2 \leqslant \frac{1}{4}\sum_{\lambda} P(\lambda|c)\left[\langle x_A^2(\lambda)\rangle + \langle p_A^2(\lambda)\rangle\right] \times \left[\langle x_B^2(\lambda)\rangle + \langle p_B^2(\lambda)\rangle\right]. \tag{I12}$$

In the case of the Category 3 states we then have

$$\begin{aligned}|\langle\widehat{a}^{\dagger}\widehat{b}\rangle|^2 &\leqslant \langle(\widehat{N}_A+\widehat{V}_A)\otimes(\widehat{N}_B+\widehat{V}_B)\rangle,\\ &= \left\langle\left(\widehat{N}_A+\tfrac{1}{2}\widehat{1}_A\right)\otimes\left(\widehat{N}_B+\tfrac{1}{2}\widehat{1}_B\right)\right\rangle,\end{aligned} \tag{I13}$$

where we note that $\widehat{N}_A + \frac{1}{2}\widehat{1}_A = \widehat{a}^{\dagger}\widehat{a} + \frac{1}{2} = (\widehat{a}\widehat{a}^{\dagger} + \widehat{a}^{\dagger}\widehat{a})/2$. This result is the same as the general result (I5) found for all Bell local states. Note also that this derivation of Eqs. ((I9),(I11)) and (I13) did not make use of the SSR. Only the presence or absence of a LHS was invoked, and whether the LHS satisfied the SSR was not used.

As will be seen in the next section, all these inequalities (I9), (I11), and (I13) can be expressed in terms of spin operator variances.

3. Correlations as spin operator inequalities: Bell local states

The inequalities (I9) and (I11) and (I13) derived above can be put into a more useful form involving *spin operators*—whose mean values and variances can be measured. From (I1) we have (see also Ref. [3])

$$\begin{aligned}|\langle\widehat{a}^{\dagger}\widehat{b}\rangle|^2 &= \langle\widehat{S}_x\rangle^2 + \langle\widehat{S}_y\rangle^2,\\ \widehat{N}_A &= \frac{1}{2}\widehat{N} - \widehat{S}_z, \qquad \widehat{N}_B = \frac{1}{2}\widehat{N} + \widehat{S}_z,\\ \widehat{S}_x^2 + \widehat{S}_y^2 + \widehat{S}_z^2 &= \frac{\widehat{N}}{2}\left(\frac{\widehat{N}}{2}+1\right).\end{aligned} \tag{I14}$$

Then we find, after some straightforward calculations and introducing the variances $\langle\Delta\widehat{S}_x^2\rangle = \langle\widehat{S}_x^2\rangle - \langle\widehat{S}_x\rangle^2$ etc., the following results for Category 1, 2, and 3 states:

$$\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle - \tfrac{1}{2}\langle\widehat{N}\rangle \geqslant 0, \quad \text{Category 1 States} \tag{I15}$$

$$\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle - \tfrac{1}{4}\langle\widehat{N}\rangle + \tfrac{1}{2}\langle\widehat{S}_z\rangle \geqslant 0, \quad \text{Category 2 States} \tag{I16}$$

$$\langle\Delta\widehat{S}_x^2\rangle + \langle\Delta\widehat{S}_y^2\rangle + \tfrac{1}{4} \geqslant 0. \quad \text{Category 3 States} \tag{I17}$$

Details are given in Appendix J. For Category 2 states with A involving the LHS then the left side would have involved $-\frac{1}{2}\langle\widehat{S}_z\rangle$.

The inequality (I16) for Category 2 states was obtained more directly *without* using the strong correlation inequalities in Secs. IV E and IV F; see Eqs. (61) and (67) and (54). Details were given in Appendix G. The inequality (I15) for Category 1 states was also derived in Refs. [14] and [3].

We note in passing that Eq. (I17) does not lead to a test for Bell nonlocality. From the Heisenberg Uncertainty Principle this inequality applies for *all* quantum states. Hence the inequality (I13) or (I17) do *not* provide a test for Bell nonlocality.

4. Weak correlation test

The quantum operator $\widehat{a}^{\dagger}\widehat{b}$ is not an observable, but from the definitions for the spin operator we can write $\widehat{a}^{\dagger}\widehat{b} = \widehat{S}_x - i\widehat{S}_y$. We have interpreted $a^{\dagger}b$ to be $S_x - iS_y$, where now S_x and S_y are observables whose mean values are definable in a LHV theory.

From (C28) and (52) we see that for Category 2 (and Category 1) states

$$\begin{aligned}\langle a^{\dagger}b\rangle &= \langle S_x\rangle - i\langle S_y\rangle,\\ &= 0,\end{aligned} \tag{I18}$$

so that

$$|\langle a^{\dagger}b\rangle|^2 = \langle S_x\rangle^2 + \langle S_y\rangle^2 = 0 \tag{I19}$$

for quantum states in Category 2 (or Category 1). This means that if

$$|\langle\widehat{a}^{\dagger}\widehat{b}\rangle|^2 > 0, \tag{I20}$$

the state cannot be either Category 1 or Category 2. This constitutes a so-called *weak correlation* test for EPR steering. However, because $|\langle\widehat{a}^{\dagger}\widehat{b}\rangle|^2 = \langle\widehat{S}_x\rangle^2 + \langle\widehat{S}_y\rangle^2$ this test is really just *equivalent* to the Bloch vector test. So no useful test for either quantum entanglement or EPR steering involving $\langle\widehat{S}_x\rangle^2 + \langle\widehat{S}_y\rangle^2$ and $\langle\widehat{N}_A\otimes\widehat{N}_B\rangle$ is established at this point. However (see Sec. I 5), it was shown that related tests can be obtained both for quantum entanglement and EPR steering.

5. Strong correlation test

Hillery and Zubairy [14] showed that for separable states (Category 1 states) that $|\langle\widehat{a}^{\dagger}\widehat{b}\rangle|^2 \leqslant \langle\widehat{a}^{\dagger}\widehat{a}\widehat{b}^{\dagger}\widehat{b}\rangle = \langle\widehat{N}_A\otimes\widehat{N}_B\rangle$. This result is also obtained here in Eq. (I9). The proof of this result was valid irrespective of whether the subsystem states $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ were local particle number SSR compliant or not (see Ref. [3] for details). The quantum result

$$\begin{aligned}|\langle\widehat{a}^{\dagger}\widehat{b}\rangle|^2 &= \langle\widehat{S}_x\rangle^2 + \langle\widehat{S}_y\rangle^2,\\ &> \langle\widehat{N}_A\otimes\widehat{N}_B\rangle,\end{aligned} \tag{I21}$$

is a *strong correlation* test for quantum *entanglement*. Hence as the numbers of bosons N_A and N_B are observables in the LHV model (and therefore the mean $\langle N_A\otimes N_B\rangle$ can be defined) we see that for Category 1 states the LHVT result

$$|\langle a^{\dagger}b\rangle|^2 \leqslant \langle N_A\otimes N_B\rangle \tag{I22}$$

applies. Thus if

$$|\langle\widehat{a}^{\dagger}\widehat{b}\rangle|^2 > \langle\widehat{N}_A\otimes\widehat{N}_B\rangle, \tag{I23}$$

we have a strong correlation test for entanglement. However, there is a *different* strong correlation test for EPR steering that applies—and which is harder to satisfy.

In the case of Category 2 states from the inequality in Eq. (I11) we see that if

$$|\langle\widehat{a}^{\dagger}\widehat{b}\rangle|^2 > \langle\widehat{N}_A\otimes\widehat{N}_B\rangle + \tfrac{1}{2}\langle\widehat{1}_A\otimes\widehat{N}_B\rangle, \tag{I24}$$

the state cannot be in Category 2 (nor in Category 1) so it must be *EPR steerable*. Thus the inequality (I24) is a *strong correlation* test for *EPR steering*. Note that the condition is harder to satisfy than the strong correlation test (I21) for entanglement since $\langle \widehat{1}_A \otimes \widehat{N}_B \rangle$ is positive, but obviously if (I24) is satisfied the state is entangled as well as being EPR steerable. If A involved the LHS then the right side would have been $\langle \widehat{N}_A \otimes (\widehat{N}_B + \frac{1}{2}\widehat{1}_B)\rangle$.

However, as these tests are just *equivalent* to the Hillery-Zubairy planar spin variance test and the generalized Hillery-Zubairy planar spin variance test, no additional test has been obtained.

APPENDIX J: CORRELATION INEQUALITIES AND SPIN OPERATORS

The inequalities (I9), (I11), and (I13) derived above can be put into a more useful form involving *spin operators*—whose mean values and variances can be measured. We use the definitions of the spin operators in Sec. IV C (see also Ref. [3])

$$\begin{aligned} |\langle \widehat{a}^\dagger \widehat{b}\rangle|^2 &= \langle \widehat{S}_x\rangle^2 + \langle \widehat{S}_y\rangle^2, \\ \widehat{N}_A &= \frac{1}{2}\widehat{N} - \widehat{S}_z, \quad \widehat{N}_B = \frac{1}{2}\widehat{N} + \widehat{S}_z, \\ \widehat{S}_x^2 + \widehat{S}_y^2 + \widehat{S}_z^2 &= \frac{\widehat{N}}{2}\left(\frac{\widehat{N}}{2} + 1\right). \end{aligned} \tag{J1}$$

g We see that

$$\begin{aligned} \langle \Delta\widehat{S}_x^2\rangle + \langle \Delta\widehat{S}_y^2\rangle &= \tfrac{1}{4}\langle (\widehat{N}_A + \widehat{N}_B)^2\rangle + \tfrac{1}{2}\langle \widehat{N}_A + \widehat{N}_B\rangle \\ &\quad - |\langle \widehat{a}^\dagger \widehat{b}\rangle|^2 - \tfrac{1}{4}\langle (\widehat{N}_B - \widehat{N}_A)^2\rangle, \end{aligned}$$

$$\begin{aligned} \langle \Delta\widehat{S}_x^2\rangle + \langle \Delta\widehat{S}_y^2\rangle &\geqslant \langle \widehat{N}_A \otimes \widehat{N}_B\rangle + \tfrac{1}{2}\langle \widehat{N}_A \otimes \widehat{1}_B\rangle \\ &\quad + \tfrac{1}{2}\langle \widehat{1}_A \otimes \widehat{N}_B\rangle - \langle \widehat{N}_A \otimes \widehat{N}_B\rangle, \\ &\geqslant \tfrac{1}{2}\langle \widehat{N}_A \otimes \widehat{1}_B\rangle + \tfrac{1}{2}\langle \widehat{1}_A \otimes \widehat{N}_B\rangle, \\ &\qquad \text{Category 1 States} \\ \langle \Delta\widehat{S}_x^2\rangle + \langle \Delta\widehat{S}_y^2\rangle &\geqslant \langle \widehat{N}_A \otimes \widehat{N}_B\rangle + \tfrac{1}{2}\langle \widehat{N}_A \otimes \widehat{1}_B\rangle \\ &\quad + \tfrac{1}{2}\langle \widehat{1}_A \otimes \widehat{N}_B\rangle \\ &\quad - \langle (\widehat{N}_A + \tfrac{1}{2}\widehat{1}_A) \otimes \widehat{N}_B\rangle, \\ &\geqslant \tfrac{1}{2}\langle \widehat{N}_A \otimes \widehat{1}_B\rangle, \qquad \text{Category 2 States} \\ \langle \Delta\widehat{S}_x^2\rangle + \langle \Delta\widehat{S}_y^2\rangle &\geqslant \langle \widehat{N}_A \otimes \widehat{N}_B\rangle + \tfrac{1}{2}\langle \widehat{N}_A \otimes \widehat{1}_B\rangle + \tfrac{1}{2}\langle \widehat{1}_A \otimes \widehat{N}_B\rangle \\ &\quad - \langle (\widehat{N}_A + \tfrac{1}{2}\widehat{1}_A) \otimes (\widehat{N}_B + \tfrac{1}{2}\widehat{1}_B)\rangle, \\ &\geqslant -\tfrac{1}{4}. \qquad \text{Category 3 States} \end{aligned} \tag{J2}$$

So we have the following:

$$\begin{aligned} \langle \Delta\widehat{S}_x^2\rangle + \langle \Delta\widehat{S}_y^2\rangle - \tfrac{1}{2}\langle \widehat{N}\rangle &\geqslant 0, \qquad \text{Category 1 States} \\ \langle \Delta\widehat{S}_x^2\rangle + \langle \Delta\widehat{S}_y^2\rangle - \tfrac{1}{4}\langle \widehat{N}\rangle + \tfrac{1}{2}\langle \widehat{S}_z\rangle &\geqslant 0, \qquad \text{Category 2 States} \\ \langle \Delta\widehat{S}_x^2\rangle + \langle \Delta\widehat{S}_y^2\rangle + \tfrac{1}{4} &\geqslant 0. \qquad \text{Category 3 States} \end{aligned} \tag{J3}$$

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